35. Fundamental Theorem of Calculus

Having developed the theory of Riemann integral, we move to the connection between differentiation and integration known under the banner Fundamental Theorem of Calculus. In Newton/Leibnitz' theory, this connection relies on the following concept:

Definition 35.1 *Let $f\colon [a,b] \to \mathbb{R}$. A function $F\colon [a,b] \to \mathbb{R}$ is is said to be an* antiderivative *of f, if F is continuous on $[a,b]$, differentiable on (a,b) and*

$$\forall x \in (a,b)\colon \quad F'(x) = f(x) \tag{35.1}$$

(Another name used for antiderivative is primitive *function.)*

We then say that, whenever a function $f\colon [a,b] \to \mathbb{R}$ admits an antiderivative F on $[a,b]$, the *Newton integral* is defined by

$$\int_a^b f(x)\mathrm{d}x := F(x)\Big|_a^b := F(b) - F(a) \tag{35.2}$$

and, in particular, f is Newton-integrable if it admits an antiderivative. (Note that if $F, G\colon [a,b] \to \mathbb{R}$ are antiderivatives of f on $[a,b]$ then $(F-G)'(x) = f(x) - f(x) = 0$ for all $x \in (a,b)$ and Rolle's Mean-Value Theorem implies that $F - G$ is constant and so the right-hand side of (35.2) is independent of the choice of the antiderivative.)

With this definition of the integral, both statements of the Fundamental Theorem of Calculus follow readily:

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x) \;\wedge\; \int_a^b F'(x)\mathrm{d}x = F(b) - F(a) \tag{35.3}$$

In Newton's integration theory, these are mathematically correct albeit not really deep statements. Indeed, the Newton integral is basically *defined* to have these properties TRUE automatically.

35.1 Differentiating the integral.

In Riemann's integration theory, the integral is defined with no *a priori* connection to the derivative and so the above become theorems that need suitable qualifiers. We start with a question: Suppose f is Riemann integrable on $[a,b]$. Then $F(x) := \int_a^x f(t)\mathrm{d}t$ is well defined for all $x \in [a,b]$. What kind of regularity can we expect from F? We already know that F is continuous but we can say a bit more:

Lemma 35.2 *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Then F defined for all $x \in (a,b]$ by $F(x) := \int_a^x f(t)\mathrm{d}t$ and by $F(a) := 0$ is Lipschitz continuous on $[a,b]$.*

Proof. Let $x, y \in [a,b]$ obey $x < y$. The additivity of the Riemann integral proved in Lemma 31.6 then shows

$$F(y) - F(x) = \int_a^y f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t = \int_x^y f(t)\mathrm{d}t \tag{35.4}$$

By Lemma 31.8, f Riemann integrable implies that f is bounded and so

$$\big|F(y) - F(x)\big| \leqslant \left|\int_x^y f(t)\mathrm{d}t\right| \leqslant \Big(\sup_{t\in[a,b]} \big|f(t)\big|\Big)|y - x| \tag{35.5}$$

Hence F is Lipschitz continuous as claimed. □

Lipschitz continuous functions are not necessarily differentiable (although they do turn out to be differentiable at "most" points). An example of such a Lipschitz continuous function is $F(x) := |x|$ for which we have

$$|x| = \int_0^x \Bigl(1_{[0,\infty)}(t) - 1_{(-\infty,0)}(t)\Bigr)\mathrm{d}t \tag{35.6}$$

The only point of non-differentiability thus coincides with a discontinuity point of the integrand. This is no coincidence in light of:

Lemma 35.3 *Let f be Riemann integrable on $[a,b]$ and set $F(x) := \int_a^x f(t)\mathrm{d}t$. Then*

$$\forall x \in (a,b)\colon \quad f \text{ continuous at } x \;\Rightarrow\; F'(x) \text{ exists} \wedge F'(x) = f(x) \tag{35.7}$$

At $x = a$ the same holds if continuity is replaced by right continuity and derivative by the right derivative, and similarly for left continuity/derivative at $x = b$.

Proof. Let $x \in [a,b)$. Then for all $y \in (x,b)$,

$$F(y) - F(x) - f(x)(y-x) = \int_x^y \bigl[f(t) - f(x)\bigr]\mathrm{d}t \tag{35.8}$$

which implies

$$\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| \leqslant \frac{1}{|y-x|}\left|\int_x^y \bigl[f(t)-f(x)\bigr]\mathrm{d}t\right| \leqslant \sup_{t\in[x,y]}\bigl|f(t)-f(x)\bigr| \tag{35.9}$$

Assuming that f is right-continuous at x, the right-hand side tends to zero as $y \to x^+$ thus proving that the right-derivative of F at x equals $f(x)$. The left-derivative at $x \in (a,b]$ is handled similarly. □

This now shows:

Theorem 35.4 (FTC I) *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then $x \mapsto \int_a^x f(t)\mathrm{d}t$ is differentiable on (a,b) — including one-sided derivatives at a and b — and*

$$\forall x \in (a,b)\colon \quad \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x) \tag{35.10}$$

Proof. This follows by applying Lemma 35.3 at all $x \in (a,b)$. □

The restriction to continuous integrand is done merely for convenience of expression. In particular, the integrand need not be continuous at the point where the integral to be differentiable. Indeed, if f only admits a limit at x then the argument (35.8–35.9) still works albeit with $f(x)$ replaced by the limit. But even the existence of the limit is not required. For instance, the function

$$f(x) := \begin{cases} 1, & \text{if } \exists n \in \mathbb{N}\colon\ x = \frac{1}{n+1}, \\ 0, & \text{else,} \end{cases} \tag{35.11}$$

has $F(x) := \int_0^x f(t)\mathrm{d}t = 0$ and so $F'(x)$ exists at all $x \in \mathbb{R}$ including $x = 0$ where the right limit of f does not exist. Another example uses the (so far undefined but standard) functions $\sin(x)$ and $\cos(x)$, the function f, defined by

$$f(x) := \frac{\mathrm{d}}{\mathrm{d}x} x^2 \sin(1/x) = \cos(1/x) - 2x\sin(1/x) \tag{35.12}$$

for $x \neq 0$ and $f(0) = 0$, is Riemann integrable on $[0,b]$ for any $b > 0$ and thus $F(x) := \int_0^x f(t)\mathrm{d}t$ is well defined for all $x \in \mathbb{R}$. Since f is continuous away from zero, we have $F'(x) = f(x)$ for all $x \neq 0$. The same applies to $G(x) := x^2\sin(1/x)$ (with $G(0) := 0$) and, by the Mean-Value Theorem and the fact that $F(0) = G(0) = 0$, we have $F(x) = G(x) = x^2\sin(1/x)$ for all $x \neq 0$. But $|F(x) - F(0)| \leqslant x^2$ implies $F'(0) = 0$ and so F is differentiable even at zero where $F'(0) = 0 = f(0)$. Yet f does not have even one-sided limits there, due to $x \mapsto \cos(1/x)$ oscillating rapidly as $x \to 0^{\pm}$.

It is worth noting that our previous characterizations of Riemann integrability tell us that integrals of Riemann integrable functions are differentiable at most points:

Corollary 35.5 *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable and let $F\colon [a,b] \to \mathbb{R}$ be defined by $F(t) := \int_a^t f(x)\mathrm{d}x$. Then*

$$\bigl\{x \in (a,b)\colon F'(x) \text{ exists } \wedge\ F'(x) = f(x)\bigr\} \tag{35.13}$$

is dense in $[a,b]$ and, in fact, the complement of a set of zero length.

Proof. That the set is the complement of a zero length set follows from Theorem 34.3 and Lemma 35.3. Such a set is automatically dense. A direct argument for density can be based on Lemma 34.1 whose proof is annotated in homework. □

Theorem 35.4 applies also to the lower limits. Indeed, we have:

Lemma 35.6 *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then*

$$\forall x \in (a,b)\colon\ \frac{\mathrm{d}}{\mathrm{d}x}\int_x^b f(t)\mathrm{d}t = -f(x) \tag{35.14}$$

Moreover, for all $g,h\colon \mathbb{R} \to [a,b]$ that are differentiable at $x \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\mathrm{d}t = f\bigl(h(x)\bigr)h'(x) - f\bigl(g(x)\bigr)g'(x) \tag{35.15}$$

We leave the proof of this lemma to a homework exercise.

35.2 Integrating the derivative.

We now move to the second part of the Fundamental Theorem of Calculus, which concerns integrals of derivatives. The precise statement is as follows:

Theorem 35.7 (FTC II) *Let $a < b$ be reals and let $F\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then*

$$F' \text{ Riemann integrable on } [a,b] \ \Rightarrow\ \int_a^b F'(x)\mathrm{d}x = F(b) - F(a) \tag{35.16}$$

Proof. Pick $\epsilon > 0$. Assuming F' to be Riemann integrable, there is $\delta > 0$ such that for all partitions Π of $[a,b]$ with $\|\Pi\| < \delta$ we have

$$\left| R(F',\Pi) - \int_a^b F'(x)\mathrm{d}x \right| < \epsilon \tag{35.17}$$

Fixing this $\delta > 0$, let $n \in \mathbb{N}$ be such that $n\delta > b-a$ and define $t_i := a + \frac{i}{n}(b-a)$ for all $i = 0, \dots, n$. By Lagrange's Mean-Value Theorem, for each $i = 1, \dots, n$,

$$\exists t_i^\star \in [t_{i-1}, t_i]\colon \quad F(t_i) - F(t_{i-1}) = F'(t_i^\star)(t_i - t_{i-1}) \tag{35.18}$$

Picking one such $t_i^\star$ in each $[t_{i-1}, t_i]$, the marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ then has mesh less than δ and obeys

$$F(b) - F(a) = \sum_{i=1}^n \bigl(F(t_i) - F(t_{i-1})\bigr) = \sum_{i=1}^n F'(t_i^\star)(t_i - t_{i-1}) = R(F',\Pi) \tag{35.19}$$

Using (35.17) it follows that

$$\left| F(b) - F(a) - \int_a^b F'(x)\mathrm{d}x \right| < \epsilon \tag{35.20}$$

As this holds for all $\epsilon > 0$, we have the conclusion of (35.16). □

The assumption that F' is Riemann integrable is not vacuous and is, in fact, a shortcoming of Riemann's theory. Indeed, we have:

Lemma 35.8 (Volterra's example) *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that F is differentiable with f' bounded yet not Riemann integrable on* $[0,1]$.

Proof. The proof is based on the notion of a "fat Cantor set." (The adjective "fat" refers to the set and not Cantor himself, of course. Per a wiki page, Volterra seems to have introduced such sets a few years before Cantor, but its first mention is due to Smith who did it even before Volterra.). This is generally defined by taking a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ and setting $C_0 := [0,1]$, $C_1 := [0,1] \smallsetminus (\frac12(1-\alpha_n), \frac12(1+\alpha_n))$ and, recursively, noting that C_n is the union of 2^n disjoint closed intervals of length ℓ_n, constructing C_{n+1} by taking a centered interval of length $\alpha_{n+1}\ell_n$ out of each of these intervals. The total length of these sets (defined by $\operatorname{length}(C_n) := 2^n\ell_n$) then obeys

$$\forall n \in \mathbb{N} \smallsetminus \{0\}\colon\ \operatorname{length}(C_n) = \prod_{i=0}^{n-1}(1-\alpha_n) \tag{35.21}$$

The right-hand side remains uniformly positive when $\sum_{n=0}^\infty \alpha_n < \infty$. In this case the set $C := \bigcap_{n\in\mathbb{N}} C_n$ is NOT zero length. Let us write I_n for the set of 2^n open intervals removed from C_n to define C_{n+1} and denote $I := \bigcup_{n\in\mathbb{N}} I_n$.

Next let $h\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that vanishes outside $(-1/2, 1/2)$ and obeys $h(0) \neq 0$. (E.g., take $h(x) := (1-4x^2)^2$ for $|x| \leqslant 1/3$ and $h(x) := 0$ otherwise.) Next let $\{\gamma_n\}_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ and, for each $n \in \mathbb{N}$ and each $(a,b) \in I_n$, we define f at $x \in (a,b)$ by

$$f(x) := \gamma_n(b-a)\, h\Bigl(\frac{1}{\gamma_n}\frac{x - \frac{a+b}{2}}{b-a}\Bigr) \tag{35.22}$$

This defines f on $[0,1] \smallsetminus C$; we then set $f(x) := 0$ at $x \in C \cup (\mathbb{R} \smallsetminus [0,1])$. It is now readily checked that, for all $n \in \mathbb{N}$ and all $(a,b) \in I_n$, the function f is differentiable on (a,b) with

$$\forall x \in (a,b)\colon\;\; f'(x) = h'\Bigl(\frac{1}{\gamma_n}\frac{x-\frac{a+b}{2}}{b-a}\Bigr) \tag{35.23}$$

On the other hand, if $x \in C$ and $z \in (a,b)$ for some $(a,b) \in I_n$, then $f(z) \neq 0$ implies $|z - \frac{a+b}{2}| \leqslant \frac{1}{2}\gamma_n(b-a)$ while $x \notin (a,b)$ gives $|x - \frac{a+b}{2}| \geqslant \frac{1}{2}(b-a)$. The triangle inequality turns this into $|x-z| \geqslant \frac{1}{2}(b-a)(1-\gamma_n)$ and (since $f(x) = 0$)

$$\forall z \in (a,b)\colon\;\; \Bigl|\frac{f(z)-f(x)}{z-x}\Bigr| \leqslant 2\frac{\gamma_n}{1-\gamma_n}. \tag{35.24}$$

Moreover, assuming $\gamma := \sup_{m\in\mathbb{N}} \gamma_m < 1$, the inequalities $|x-z| \geqslant \frac{1}{2}(b-a)(1-\gamma)$ and $|x-z| < \delta$ imply $(b-a) \leqslant 2(1-\gamma)^{-1}\delta$. If $z \in (a,b)$ for some $(a,b) \in I_n$ and $|x-z| < \delta$ for some $x \in C$, then $f(z) \neq 0$ forces n to be so large that $2^{-n}\mathrm{length}(C) < 4\delta$. Putting these observations together we conclude

$$\lim_{n\to\infty} \gamma_n = 0 \;\Rightarrow\; \forall x \in C\colon\;\; f'(x) \text{ exists} \,\wedge\, f'(0) = 0. \tag{35.25}$$

In particular, once $\gamma_n \to 0$ the function f is everywhere differentiable with f' bounded.

By the way C is constructed, for each $x \in C$ and each $\delta > 0$, there exists an interval $(a,b) \in I$ such that $(a,b) \subseteq (x-\delta, x+\delta)$. From (35.23) we thus get

$$\forall x \in C\colon\;\; \limsup_{z\to x} f'(z) \geqslant \sup_{z\in\mathbb{R}} h'(z) > 0 \tag{35.26}$$

where the positivity comes from the fact that h is non-constant. Combined with (35.25) this implies

$$C \subseteq \bigl\{x \in \mathbb{R}\colon f' \text{ NOT continuous at } x\bigr\} \tag{35.27}$$

As C fails to be of zero length, so does the set on the right. By Theorem 34.3, f' is NOT Riemann integrable on $[0,1]$ $\square$

We remark that Volterra's example has been a source of motivation for the creation of Lebesgue's theory of integration; indeed, in this theory an everywhere differentiable function with a bounded derivative does satisfy the conclusion of (35.16). (This is because the derivative of an everywhere differentiable function is necessarily *measurable*, which is the type of regularity required by the Lebesgue integral.) Voltera's function is Lipschitz continuous and so we have shown that there are differentiable Lipschitz functions that do NOT arise as Riemann integrals. (As an aside we note that all such functions do arise as Lebesgue integrals.)

Theorem 35.4 and 35.7 give us the precise versions of (35.3) in Riemann's theory. In summary, they say that

- derivative inverts Riemann integrals of all continuous functions, and
- Riemann integral inverts all Riemann integrable derivatives.

Neither inversion is thus perfect, unlike for the Newton integral. Also Lebesgue's integral requires continuity of integrand in order for the integral to be inverted by differentiation. For integrals of derivatives the Lebesgue integral does fare slightly better as it inverts all bounded derivatives. Unfortunately, there exists an everywhere-differentiable function whose (unbounded) derivative is not Lebesgue integrable and so even there the connection requires additional qualifiers. We will return to this briefly in Section 37.

35.3 Applications.

The Fundamental Theorem of Calculus, albeit somewhat restricted in Riemann's theory, serves as a basic tool for computing integrals. However, it underlies also other tools that are generally used to convert one integral to another. Here is one frequently used:

Corollary 35.9 (Integration by parts) *Suppose $f, g\colon [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b) such that f' and g' — with values at a and b chosen arbitrarily — are Riemann integrable on $[a,b]$. Then*

$$\int_a^b f'(x)g(x)\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b g'(x)f(x)\mathrm{d}x \tag{35.28}$$

Proof. The Product Rule for derivative shows that $f \cdot g$ is differentiable and, under our condition, $(f \cdot g)' = f'g + g'f$ is Riemann integrable. Theorem 35.7 shows

$$\int_a^b \bigl[f'(x)g(x) + g'(x)f(x)\bigr]\mathrm{d}x = \int_a^b (f \cdot g)'(x)\mathrm{d}x = f(b)g(b) - f(a)g(a) \tag{35.29}$$

Since $f'g$ and $g'f$ are individually Riemann integrable, the integral on the left-hand side can be written as the sum of two Riemann integrals. □

Another standard method for converting one integral to another is:

Corollary 35.10 (Substitution Rule) *Let $c < d$ and $a < b$ be reals and assume $f\colon [c,d] \to \mathbb{R}$ and $\varphi\colon [a,b] \to [c,d]$ are functions such that:*

1. *φ is continuous on $[a,b]$ and differentiable on (a,b),*
2. *f is continuous on $[c,d]$,*
3. *$(f \circ \varphi) \cdot \varphi'$ is Riemann integrable on $[a,b]$.*

Then

$$\int_{\varphi(a)}^{\varphi(b)} f(t)\mathrm{d}t = \int_a^b f\bigl(\varphi(x)\bigr)\varphi'(x)\mathrm{d}x \tag{35.30}$$

Proof. Since f is continuous, $F(x) := \int_c^x f(t)\mathrm{d}t$ is well defined and, by Theorem 35.4, obeys $F'(t) = f(t)$ for all $t \in (c,d)$. Hence also the derivative of $F \circ \varphi$ equals the product $(f \circ \varphi) \cdot \varphi'$. Theorem 35.7 then equates both sides of (35.30) with $F(\varphi(b)) - F(\varphi(a))$. □

A more substantive application of FTC is the content of:

Theorem 35.11 (Taylor theorem with remainder) *Let $a < b$ be reals and $f\colon (a,b) \to \mathbb{R}$ an $(n+1)$-times differentiable function, for some $n \in \mathbb{N}$. Assume $f^{(n+1)}$ is Riemann integrable on any closed subinterval of (a,b). Then*

$$\forall x, x_0 \in (a,b)\colon \quad f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(z)(x-z)^n\mathrm{d}z \tag{35.31}$$

Proof. We will prove this by induction on n. For $n = 0$ the statement (35.31) is just Theorem 35.7 (which requires only that f' is Riemann integrable). Assume therefore that the statement holds for some n and let $f\colon (a,b)\to\mathbb{R}$ be now $(n+2)$-times differentiable with $f^{(n+2)}$ Riemann integrable. Abbreviating the polynomial on the right of (35.31) as $P_n(x)$, from the statement for n we then have

$$f(x) = P_n(x) + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(z)(x-z)^n \mathrm{d}z \tag{35.32}$$

Integration by parts; namely, Corollary 35.9 with $g(z) := \frac{1}{n+1}(x-z)^{n+1}$ then shows

$$\begin{aligned}
&\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(z)(x-z)^n \mathrm{d}z \\
&\qquad= \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(z)g'(z)\mathrm{d}z \\
&\qquad= \frac{1}{n!}f^{(n+1)}(z)g(z)\Big|_{x_0}^{x} - \frac{1}{n!}\int_{x_0}^{x} f^{(n+2)}(z)g(z)\mathrm{d}z \\
&\qquad= \frac{1}{(n+1)!}f^{(n+1)}(x_0)(x-x_0)^{n+1} + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(z)(x-z)^{n+1}\mathrm{d}z
\end{aligned} \tag{35.33}$$

Noting that the first term on the right equals $P_{n+1}(x) - P_n(x)$, we have proved (35.31) for n replaced by $n+1$. By induction, the claim holds for all $n\in\mathbb{N}$. □

The statement (35.31) should be compared with the pointwise version of Taylor's theorem in which the error $f(x) - P_n(x)$ takes the form $\frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1}$ for some ξ between x_0 and x. This term usually fares similarly when uniform estimates on the error are required but has the disadvantage of being dependent on an unknown intermediate point ξ. The error in (35.31) is expressed as an explicit integral and is thus better suited when further manipulations with the error term are required.