34. Lebesgue's characterization of Riemann integrability

The previous section gave a number of sufficient conditions for Riemann integrability but the only necessary condition that we have had so far is boundedness. The sufficient conditions all had to do with the structure of the continuity set of the function under integral. In this section we will formulate a condition on the continuity set that is both necessary and sufficient for Riemann integrability. This will allow us to characterize the class of Riemann integrable functions completely.

34.1 Statement and preliminaries.

We have seen that a function can be Riemann integrable while being discontinuous at a set of cardinality of the continuum. However, the sets of discontinuities cannot contain non-degenerate intervals :

Lemma 34.1 *Let $f\colon [a,b]\to\mathbb{R}$ be bounded. Then*

$$f \text{ RI } \Rightarrow \big\{x\in[a,b]\colon f \text{ continuous at } x\big\} \text{ dense in } [a,b]. \tag{34.1}$$

We leave the proof of this lemma to a homework problem while noting that the Dirichlet function, $1_{\mathbb{Q}}$, is an example of a function which is discontinuous at all points and thus fails to be Riemann integrable. As the example in (33.10) shows, the complement of (34.1) can be dense in $[a,b]$ with f Riemann integrable on $[a,b]$. As it turns out, the criterion that decides whether the discontinuity set is too large is based neither on cardinality nor on (metric-space) topology; instead, it is based on measure.

Definition 34.2 *We say that a set $A\subseteq\mathbb{R}$ is of* zero length *if*

$$\forall\epsilon>0\,\exists\{(a_i,b_i)\}_{i\in\mathbb{N}} \text{ intervals}\colon\ A\subseteq\bigcup_{i\in\mathbb{N}}(a_i,b_i)\ \wedge\ \sum_{i=0}^{\infty}(b_i-a_i)<\epsilon \tag{34.2}$$

We then have:

Theorem 34.3 (Lebesgue's characterization of Riemann integrability) *Let $a<b$ be reals and $f\colon[a,b]\to\mathbb{R}$ a bounded function. Then*

$$f \text{ RI on } [a,b] \Leftrightarrow \big\{x\in[a,b]\colon f \text{ NOT continuous at } x\big\} \text{ zero length} \tag{34.3}$$

Before we delve into the proof, we note:

Lemma 34.4 *The statement in* (34.2) *is equivalent to that in which the open intervals $\{(a_i,b_i)\}_{i\in\mathbb{N}}$ are replaced by closed intervals $\{[a_i,b_i]\}_{i\in\mathbb{N}}$.*

Proof. Since $(a_i,b_i)\subseteq[a_i,b_i]$, if (34.2) holds with open intervals then it holds with closed intervals. Conversely, assuming that (34.2) holds for a given $\epsilon>0$ with intervals $\{[a_i,b_i]\}_{i\in\mathbb{N}}$. Now set

$$a_i' := a_i-\epsilon 2^{-i}\ \wedge\ b_i' := b_i+\epsilon 2^{-i} \tag{34.4}$$

Then $A\subseteq\bigcup_{i\in\mathbb{N}}[a_i,b_i]\subseteq\bigcup_{i\in\mathbb{N}}(a_i',b_i')$ and

$$\sum_{i=0}^{\infty}(b_i'-a_i')=\sum_{i=0}^{\infty}(b_i-a_i)+\sum_{i=0}^{\infty}2\epsilon 2^{-i}<\epsilon+4\epsilon=5\epsilon \tag{34.5}$$

and so (34.2) holds with open intervals for each $\epsilon > 0$ as well. □

For the proof, we also need:

Lemma 34.5 *Given a bounded* $f\colon \mathbb{R} \to \mathbb{R}$ *with* $\mathrm{Dom}(f) = \mathbb{R}$, *define* $M_f, m_f\colon \mathbb{R} \to \mathbb{R}$ *by*

$$M_f(x) := \inf_{\delta>0} \sup_{y\in(x-\delta,x+\delta)} f(y) \tag{34.6}$$

and

$$m_f(x) := \sup_{\delta>0} \inf_{y\in(x-\delta,x+\delta)} f(y) \tag{34.7}$$

Then we have:

(1) $\forall x \in \mathbb{R}\colon\ m_f(x) \leqslant f(x) \leqslant M_f(x)$,
(2) $\forall x \in \mathbb{R}\colon\ f$ continuous at $x \Leftrightarrow M_f(x) = m_f(x)$
(3) $\forall x \in \mathbb{R}\,\forall \delta > 0\colon$

$$\begin{aligned}&\operatorname{osc}\bigl(f,[x-\delta,x]\bigr) + \operatorname{osc}\bigl(f,[x,x+\delta]\bigr)\\ &\qquad\qquad \geqslant \operatorname{osc}\bigl(f,[x-\delta,x+\delta]\bigr) \geqslant M_f(x) - m_f(x)\end{aligned} \tag{34.8}$$

(4) $\forall x \in \mathbb{R}\colon$

$$\lim_{\delta\to0^+} \operatorname{osc}\bigl(f,[x-\delta,x+\delta]\bigr) = M_f(x) - m_f(x) \tag{34.9}$$

Proof. (1) is checked immediately from the definitions. For (2) we note that continuity is equivalent to for each $\epsilon > 0$ there being $\delta > 0$ such that $f(x) - \epsilon \leqslant f(y) \leqslant f(x) + \epsilon$ being true for all $y \in (x-\delta, x+\delta)$. This shows that continuity is equivalent to

$$\forall \epsilon > 0\,\exists \delta > 0\colon\ \sup_{y\in(x-\delta,x+\delta)} f(y) - \inf_{y\in(x-\delta,x+\delta)} f(y) \leqslant \epsilon \tag{34.10}$$

which is thus equivalent to $M_f(x) - m_f(x) = 0$.

As to (3), since both sides are unchanged if a constant is added to f, we may assume that $f(x) = 0$. Then

$$\sup_{y\in[x-\delta,x]} f(x) + \sup_{y\in[x-\delta,x]} f(x) \geqslant \sup_{y\in[x-\delta,x+\delta]} f(y) \tag{34.11}$$

and

$$\inf_{y\in[x-\delta,x]} f(x) + \inf_{y\in[x-\delta,x]} f(x) \leqslant \inf_{y\in[x-\delta,x+\delta]} f(y) \tag{34.12}$$

Subtracting both sides and invoking (32.20) we get

$$\begin{aligned}&\operatorname{osc}\bigl(f,[x-\delta,x]\bigr) + \operatorname{osc}\bigl(f,[x,x+\delta]\bigr)\\ &\qquad\qquad \geqslant \sup_{y\in[x-\delta,x+\delta]} f(y) - \inf_{y\in[x-\delta,x+\delta]} f(y) \geqslant M_f(x) - m_f(x)\end{aligned} \tag{34.13}$$

thus proving (34.8).

For (4) we invoke (32.20) to cast the oscillation in (34.9) as the difference of the supremum and the infimum of f over $[x-\delta, x+\delta]$ and note that the infimum/supremum over $\delta > 0$ in (34.6–34.7) can be replaced by the limit as $\delta \to 0^+$. (This also shows that we can write the closed interval in (34.6–34.7) with no difference to the resulting quantity.) □

34.2 Zero length implies Riemann integrability.

We will now move to the proof of the main statement, proving separately the implications $\Rightarrow$ and $\Leftarrow$ in (34.3). We start with:

Proof of $\Leftarrow$ *in* (34.3). Let $f\colon [a,b] \to \mathbb{R}$ be bounded and extend it by constants to a function on all of $\mathbb{R}$. Suppose that the set of discontinuities of f is of zero length. This means that, given $\epsilon > 0$ there are open intervals $\{J_i\}_{i\in\mathbb{N}}$ such that

$$\bigl\{x \in [a,b]\colon M_f(x) > m_f(x)\bigr\} \subseteq \bigcup_{i\in\mathbb{N}} J_i \ \wedge\ \sum_{i=0}^{\infty} \text{length}(J_i) < \frac{\epsilon}{1+\sup|f|} \tag{34.14}$$

At all continuity points $x \in [a,b]$ of f, Lemma 34.5(3) shows that the set int

$$\delta_x := \frac{1}{2}\sup\Bigl\{\delta \in (0,1]\colon \text{osc}\bigl(f, [x-\delta_x, x+\delta_x]\bigr) < \frac{\epsilon}{b-a}\Bigr\} \tag{34.15}$$

is non-empty and the supremum is thus a number in $(0,1]$. In particular, we have

$$\text{osc}\bigl(f, [x-\delta_x, x+\delta_x]\bigr) < \frac{\epsilon}{b-a} \tag{34.16}$$

The collection of open intervals

$$\Bigl\{(x-\delta_x, x+\delta_x)\colon x \in [a,b] \ \wedge\ M_f(x) = m_f(x)\Bigr\} \cup \{J_i\colon i \in \mathbb{N}\} \tag{34.17}$$

then covers $[a,b]$ which is compact and so, by the Heine-Borel Theorem, the collection (34.17) contains a finite subcollection still covering $[a,b]$. This can be phrased by saying that there are $m,n \in \mathbb{N}$ and points $x_0,\dots,x_m \in \mathbb{R}$ such that

$$[a,b] \subseteq \bigcup_{i=0}^{n} J_i \cup \bigcup_{k=0}^{m} (x_k - \delta_{x_k}, x_k + \delta_{x_k}) \tag{34.18}$$

Let $\Pi = \{t_i\}_{i=0}^N$ be the partition consisting of a and b and all the endpoints of the intervals on the right of (34.18) that lie in $[a,b]$. Each interval $[t_{i-1}, t_i]$ must then lie in the closure of at least one of the intervals on the right of (34.18). Denote

$$I := \Bigl\{i = 1,\dots,N\colon \bigl(\exists k = 0,\dots,m\colon [t_{i-1},t_i] \subseteq [x_k - \delta_{x_k}, x_k + \delta_{x_k}]\bigr)\Bigr\} \tag{34.19}$$

and, for each $j = 0,\dots,n$, let

$$K_j := \bigl\{i = 1,\dots,N\colon\ i \notin I \ \wedge\ [t_{i-1},t_i] \subseteq \overline{J}_j\bigr\} \tag{34.20}$$

Note that the disjoint union $\bigcup_{i\in K_j}(t_{i-1},t_i)$ is contained in J_j and so

$$\sum_{i\in K_j}(t_i - t_{i-1}) \leqslant \text{length}(J_j) \tag{34.21}$$

by an elementary telescoping argument. With these in hand, (34.16) gives

$$\forall i \in I\colon\ \text{osc}\bigl(f, [t_{i-1},t_i]\bigr) < \frac{\epsilon}{b-a} \tag{34.22}$$

while (34.14) along with (34.21) show

$$\sum_{i\notin I}(t_i - t_{i-1}) = \sum_{j=0}^{m}\sum_{i\in K_j}(t_i - t_{i-1}) \leqslant \sum_{j=0}^{m}\text{length}(J_j) < \frac{\epsilon}{1+\sup|f|} \tag{34.23}$$

Hence we get

$$\begin{aligned}\sum_{i=1}^{n} &\operatorname{osc}\bigl(f,[t_{i-1},t_i]\bigr)(t_i-t_{i-1})\\ &=\sum_{i\in I}\operatorname{osc}\bigl(f,[t_{i-1},t_i]\bigr)(t_i-t_{i-1})+\sum_{i\notin I}\operatorname{osc}\bigl(f,[t_{i-1},t_i]\bigr)(t_i-t_{i-1})\\ &\leqslant \frac{\epsilon}{b-a}\sum_{i\in I}(t_i-t_{i-1})+2\sup|f|\sum_{i\notin I}(t_i-t_{i-1})\\ &\leqslant \epsilon+2\sup|f|\frac{\epsilon}{1+\sup|f|}\leqslant 3\epsilon\end{aligned} \tag{34.24}$$

Using Theorem 32.9, f is thus Riemann integrable. □

As a consequence of the implication we just proved, we get:

Corollary 34.6 *Let $f\colon [a,b]\to\mathbb{R}$ be bounded and has at most countably many discontinuity points. Then f is Riemann integrable.*

Proof. It suffices to show that any finite or countable set $A\subseteq\mathbb{R}$ is zero length. Pick $\epsilon>0$ and write A into a sequence $\{x_n\}_{n\in\mathbb{N}}$. Define $I_n := (x_n-\epsilon 2^{-n}, x_n+\epsilon 2^{-n})$. Then $A\subseteq \bigcup_{n\in\mathbb{N}} I_n$ and yet

$$\sum_{n\in\mathbb{N}}\operatorname{length}(I_n)=\epsilon\sum_{n\in\mathbb{N}}2\cdot 2^{-n}=4\epsilon. \tag{34.25}$$

Hence A is of zero length. □

34.3 Riemann integrability implies zero length.

We will now continue to prove that the clause on the right of (34.3) is also sufficient for Riemann integrability.

Proof of ⇒ in (34.3). Suppose $f\colon[a,b]\to\mathbb{R}$ is Riemann integrable. We will extend f to a function $\mathbb{R}\to\mathbb{R}$ by $f(x):=f(a)$ for $x<a$ and $f(x):=f(b)$ for $x>b$. All discontinuity points of this extension then still lie in $[a,b]$.

Pick $\epsilon>0$. Then Theorem 32.9 ensures that for each $n\in\mathbb{N}$ there exists a partition $\Pi_n=\{t_i^n\}_{i=0}^{m(n)}$ of $[a,b]$ such that

$$\sum_{i=1}^{m(n)}\operatorname{osc}\bigl(f,[t_{i-1}^n,t_i^n]\bigr)(t_i^n-t_{i-1}^n)<\epsilon 4^{-n} \tag{34.26}$$

(Here t_i^n is a notation for the partition point, not the n-th power or t_i.) Let

$$I_n:=\Bigl\{i=1,\dots,m(n)\colon \operatorname{osc}\bigl(f,[t_{i-1}^n,t_i^n]\bigr)>2^{-n}\Bigr\} \tag{34.27}$$

We claim that then

$$\bigl\{x\in[a,b]\colon M_f(x)>m_f(x)\bigr\}\subseteq\bigcup_{n\in\mathbb{N}}\bigcup_{i\in I_n}[t_{i-1}^n,t_i^n] \tag{34.28}$$

Indeed, if $M_f(x)>m_f(x)$ then there is $n\in\mathbb{N}$ such that $M_f(x)-m_f(x)\geqslant 2^{-n+1}$ and so there exists $i\in\{1,\dots,m(n)\}$ such that $x\in[t_{i-1}^n,t_i^n]$. Assuming first that $x\neq t_{i-1}^n,t_i^n$, (34.8) then shows that, for each $\delta > 0$,

$$\max\Big\{\operatorname{osc}\big(f,[x-\delta,x]\big),\operatorname{osc}\big(f,[x,x+\delta]\big)\Big\} \geqslant \frac{1}{2}\big[M_f(x)-m_f(x)\big] > 2^{-n} \tag{34.29}$$

and, since x is an interior point of $[t^n_{i-1},t^n_i]$, also

$$\operatorname{osc}\big(f,[t^n_{i-1},t^n_i]\big) > 2^{-n} \tag{34.30}$$

by choosing $\delta < \min\{x-t^n_{i-1},t^n_i-x\}$. If $x = t^n_i$ then the same argument only gives

$$\max\Big\{\operatorname{osc}\big(f,[t^n_{i-1},t^n_i]\big),\operatorname{osc}\big(f,[t^n_i,t^n_{i+1}]\big) > 2^{-n} \tag{34.31}$$

provided that $i < m(n)$, and so at least one of the two oscillations exceeds 2^{-n}. When $i = m(n)$ we get (34.30) again because f is constant on $[b,\infty)$ and so its oscillation over any interval contained therein vanishes. The case when $x = t^n_{i-1}$ is treated analogously and so, in all cases, we have demonstrated the existence of an $i \in I_n$ such that $x \in [t^n_{i-1},t^n_i]$. This proves the inclusion (34.28).

In light of Lemma 34.4, it suffices to control the total length of the intervals

$$\bigcup_{n\in\mathbb{N}} \big\{[t^n_{i-1},t^n_i]\colon i \in I_n\big\} \tag{34.32}$$

First, using $\operatorname{osc}\big(f,[t^n_{i-1},t^n_i]\big)/2^{-n} \geqslant 1$ for each $i \in I_n$ we get

$$\begin{aligned}\sum_{i\in I_n}(t^n_i-t^n_{i-1}) &\leqslant \sum_{i\in I_n}\frac{\operatorname{osc}\big(f,[t^n_{i-1},t^n_i]\big)}{2^{-n}}(t^n_i-t^n_{i-1}) \\ &\leqslant 2^n\sum_{i=1}^{m(n)}\operatorname{osc}\big(f,[t^n_{i-1},t^n_i]\big)(t^n_i-t^n_{i-1}) < 2^n\epsilon 4^{-n} = \epsilon 2^{-n}\end{aligned} \tag{34.33}$$

(The argument for the first $\leqslant$ goes by the name *Markov inequality*.) Summing this over $n \in \mathbb{N}$ then shows

$$\sum_{n=0}^{\infty}\sum_{i\in I_n}(t^n_i-t^n_{i-1}) \leqslant \sum_{n=0}^{\infty}\epsilon 2^{-n} = 2\epsilon \tag{34.34}$$

As this holds for all $\epsilon > 0$, the set on the left of (34.28), which by Lemma 34.5 coincides with the discontinuity set of f, is of zero length. □

We finish with two remarks. First, the proof of Theorem 34.3 is considerably easier once the apparatus of measure theory and Lebesgue integral becomes available. Indeed, in this case we show that (assuming f bounded),

$$\int_{[a,b]} m_f \mathrm{d}\lambda = \underline{\int_a^b} f(x)\mathrm{d}x \leqslant \overline{\int_a^b} f(x)\mathrm{d}x = \int_{[a,b]} M_f \mathrm{d}\lambda \tag{34.35}$$

where the intervals on extreme ends of this inequality are Lebesgue integrals and λ is the so called Lebesgue measure. Hence we get

$$\underline{\int_a^b} f(x)\mathrm{d}x = \overline{\int_a^b} f(x)\mathrm{d}x \;\Leftrightarrow\; \int_{[a,b]}(M_f-m_f)\mathrm{d}\lambda \tag{34.36}$$

and, since $M_f - m_f \geqslant 0$, the existence of the Darboux integral is equivalent to $M_f = m_f$ being TRUE except on a set of zero length (or, technically, a λ-null set).

This being said, the proof we presented above is quite hard and so we caution the reader not to use Theorem 34.3 as a means to prove Riemann integrability in situations that can be handled by considerably simpler arguments. (This applies, in particular, to all of the sufficient conditions for Riemann integrability discussed earlier.)