## 33. Sufficient conditions for Riemann integrability

As noted above, one principal novelty of Riemann's approach to integration is to make the existence of the integral a regularity property of the function in its own right. However, this still leaves us with the need for easily checkable sufficient conditions for Riemann integrability. In this section we go over a list of progressively more demanding sufficient conditions. This will reveal ideas that will play an important role in our full characterization of Riemann integrability in the next section.

### 33.1 Continuous functions and variations thereof.

One of the earlier attempts to make the concept of integral well defined was put forward by Cauchy, who insisted on working with continuous integrands. It is reassuring that Cauchy's treatment becomes subsumed by Riemann's:

**Lemma 33.1** *Let $f\colon [a,b]\to\mathbb{R}$. Then*

$$f \text{ continuous} \;\Rightarrow\; f \text{ Riemann integrable} \tag{33.1}$$

*Proof.* By the Bolzano-Weierstrass Theorem, a continuous $f\colon [a,b]\to\mathbb{R}$ is automatically bounded and uniformly continuous. This means that

$$\forall\epsilon>0\,\exists\delta>0\,\forall s,t\in[a,b]\colon\; 0<t-s<\delta \;\Rightarrow\; \operatorname{osc}\bigl(f,[s,t]\bigr)<\frac{\epsilon}{b-a} \tag{33.2}$$

Picking such a $\delta>0$, ti follows that for any partition $\Pi=\{t_i\}_{i=1}^n$ of $[a,b]$,

$$\|\Pi\|<\delta \;\Rightarrow\; \sum_{i=1}^n \operatorname{osc}\bigl(f,[t_{i-1},t_i]\bigr)(t_i-t_{i-1})<\frac{\epsilon}{b-a}\sum_{i=1}^n(t_i-t_{i-1})=\epsilon \tag{33.3}$$

As one such a partition can definitely be constructed, Theorem 32.9 implies that $f$ is Riemann integrable. □

The previous lemma notwithstanding, Riemann integral does not at all require the integrand to be continuous. For instance, we have:

**Lemma 33.2** *Let $f\colon [a,b]\to\mathbb{R}$ be bounded with a finite number of discontinuities. Then $f$ is Riemann integrable on $[a,b]$ and the values of $f$ at the discontinuity points are immaterial for the Riemann integral.*

*Proof.* Fix $\epsilon>0$ and let $x_1,\dots,x_n$ enumerate the discontinuities of $f$. Pick $\delta'$ with

$$0<\delta'<\min\Bigl\{\frac12\min_{0\leqslant i<j\leqslant n}|x_i-x_j|,\,\frac1n\frac{\epsilon}{1+\sup|f|}\Bigr\} \tag{33.4}$$

where $\sup|f|:=\sup_{x\in[a,b]}|f(x)|$. Then $f$ is continuous on

$$A:=[a,b]\smallsetminus\bigcup_{i=0}^n(x_i-\delta',x_i+\delta') \tag{33.5}$$

Since $A$ is closed and, being a subset of a compact set, compact, the Bolzano-Weierstrass Theorem still applies to give us a $\delta''>0$ such that

$$\forall s,t\in A\colon\; 0<t-s<\delta''\;\wedge\;[s,t]\subseteq A \;\Rightarrow\; \operatorname{osc}\bigl(f,[s,t]\bigr)<\frac{\epsilon}{b-a} \tag{33.6}$$

Now pick $m \in \mathbb{N}$ such that $m\delta'' > b-a$ and consider the partition $\Pi = \{t_i\}_{i=1}^N$ consisting of the points $x_i - \epsilon$ and $x_i + \epsilon$ for $i = 1, \dots, n$ and the points of the form $a + j/N$ indexed by $j = 0, \dots, m$ that lie in $A$. For each $i = 1, \dots, N$, the interval $[t_{i-1}, t_i]$ then coincides with one of $[x_k - \delta', x_k + \delta']$ (because, by our choice of $\delta'$, these intervals are disjoint from each other) or is an interval contained in $A$ of length less than $\delta''$. Denote

$$I := \{i = 1, \dots, N \colon [t_{i-1}, t_i] \subseteq A\} \tag{33.7}$$

Then (33.6) gives

$$\sum_{i\in I} \operatorname{osc}\bigl(f, [t_{i-1}, t_i]\bigr)(t_i - t_{i-1}) < \frac{\epsilon}{b-a}\sum_{i=1}^{n}(t_i - t_{i-1}) < \frac{\epsilon}{b-a}\sum_{i=1}^{N}(t_i - t_{i-1}) < \epsilon \tag{33.8}$$

while $\delta' < n^{-1}\epsilon/(1+\sup|f|)$ and $\operatorname{osc}(f, E) \leqslant 2\sup|f|$ and $N \smallsetminus |I| \leqslant n$ give

$$\sum_{i\notin I} \operatorname{osc}\bigl(f, [t_{i-1}, t_i]\bigr)(t_i - t_{i-1}) \leqslant n(2\sup|f|)(2\delta') < \frac{4n\epsilon\sup|f|}{n(1+\sup|f|)} \leqslant 4\epsilon \tag{33.9}$$

Putting (33.8–33.9) together and invoking (32.21) yields $U(f,\Pi) - L(f,\Pi) < 5\epsilon$. Theorem 32.9 implies that $f$ is Riemann integrable. □

### 33.2 More intricate examples.

Thinking about the previous proof, a finite number of discontinuities is not at all the limit of what the Riemann integral is able to take. For instance, writing all of the rationals into a sequence $\{q_n\}_{n\in\mathbb{N}}$, the function

$$f(x) := \begin{cases} \frac{1}{n+1}, & \text{if } x = q_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases} \tag{33.10}$$

is Riemann integrable even though it is discontinuous at each rational. This follows from the following more general fact:

**Lemma 33.3** *Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded and such that $\lim_{z\to x} f(z)$ exists for all $x \in [a,b]$. Then $f$ is Riemann integrable on $[a,b]$.*

While the proof of this is instructive, we can directly shoot for a stronger result than this and instead state and prove:

**Lemma 33.4** *Let $f\colon [a,b] \to \mathbb{R}$ be bounded and with no discontinuities of the second kind on $(a,b)$. Then $f$ is Riemann integrable on $[a,b]$.*

*Proof.* Recall that a function has a discontinuity of a second kind at $x \in [a,b]$ if a least one of the one-sided limits of $f$ does not exist. So our $f$ has both left and right limits $f(x^-)$ and $f(x^+)$ at each $x \in (a,b)$. The key point is to realize that then

$$\forall \eta > 0\colon \quad \Bigl\{x \in (a,b)\colon \operatorname{diam}(\{f(x), f(x^+), f(x^-)\}) > \eta\Bigr\} \text{ is finite} \tag{33.11}$$

where, we recall, for any $A \subseteq \mathbb{R}$ we put $\operatorname{diam}(A) := \sup\{|x-y| \colon x, y \in A\}$. Leaving the proof of (33.11) to the reader, pick $\epsilon > 0$ and let $x_1, \dots, x_n$ enumerate the set in (33.11) for $\eta := \frac{1}{2}\epsilon/(b-a)$. Picking again $\delta' > 0$ such that $\delta' < n^{-1}\epsilon/(1+\sup|f|)$, let $A$ be as in (33.5). Then

$$\forall z \in A\colon \quad \operatorname{diam}\bigl(\{f(z), f(z^+), f(z^-)\}\bigr) \leqslant \eta = \frac{1}{2}\frac{\epsilon}{b-a} \tag{33.12}$$

along with the existence of the right/left limits ensures

$$\forall z \in A\, \exists \delta_z > 0\colon \ \operatorname{osc}\bigl(f, [z-2\delta_z, x+2\delta_z]\bigr) < \frac{\epsilon}{b-a} \tag{33.13}$$

The intervals $\{(z-\delta_z, z+\delta_z)\colon z \in A\}$ cover $A$ and, since $A$ is compact, the Heine-Borel Theorem gives us $z_1, \dots, z_m$ such that

$$A \subseteq \bigcup_{i=1}^{m} (z_i - \delta_{z_i}, z_i + \delta_{z_i}) \tag{33.14}$$

Let $\Pi = \{t_i\}_{i=0}^N$ be the partition that consists of $a$ and $b$, all the points of the form $x_i \pm \delta'$, $i = 1, \dots, n$, that lie in $[a, b]$ and also all the points of the form $z_i \pm \delta_{z_i}$, $i = 1, \dots, m$, that lie in $[a, b]$. Each interval interval $[t_{i-1}, t_i]$ in the partition is then contained either in $[x_j - \delta', x_j + \delta']$ for some $j = 1, \dots, n$ and then $\operatorname{osc}(f, [t_{i-1}, t_i]) \leqslant 2\sup|f|$, or lies in some $(z_j - 2\delta_{z_j}, z + 2\delta_{z_j})$ and then $\operatorname{osc}(f, [t_{i-1}, t_i]) < \epsilon/(b-a)$. The same calculation as in the previous proof then gives (33.9) and thus proves Riemann integrability of $f$ on $[a, b]$. □

As part of the proof, we have actually shown:

**Corollary 33.5** *Let $f\colon [a,b] \to \mathbb{R}$ has has no discontinuities of the second kind. Then*

$$\Bigl\{x \in [a,b]\colon f \text{ NOT continuous at } x\Bigr\} \text{ is finite or countable} \tag{33.15}$$

*Proof.* This follows by taking the union of the sets in (33.11) for $\eta \in \{\frac{1}{n+1}\colon n \in \mathbb{N}\}$ and noting that this union exhaust the set of all discontinuities. □

As we will see in Theorem 34.3, under boundedness of $f$ even just (33.15) is sufficient to conclude Riemann integrability of $f$. The proof of this is already quite close to the proof of Theorem 34.3 itself and so we will not present that here. However, once the cardinality of the set of discontinuities is uncountable, the situation is more complicated. The extreme example is:

**Lemma 33.6** *The Dirichlet function $1_{\mathbb{Q}}$ is not integrable on any bounded closed interval.*

*Proof.* Since $\operatorname{osc}(1_{\mathbb{Q}}, [s,t]) = 1$ for all real $s < t$, for any partition $\{t_i\}_{i=0}^n$ of $[a, b]$,

$$\sum_{i=1}^{n} \operatorname{osc}\bigl(1_{\mathbb{Q}}, [t_{i-1}, t_i]\bigr)(t_i - t_{i-1}) = b - a \tag{33.16}$$

thus showing that (32.22) is FALSE. □

However, as it turns out, the cardinality is not the primary culprit here as our next example shows:

**Lemma 33.7** *Recall that the Cantor ternary set is defined as*

$$C := \left\{ \sum_{i=0}^{\infty} \frac{2\sigma_i}{3^{i+1}}\colon \sigma \in \{0,1\}^{\mathbb{N}} \right\} \tag{33.17}$$

*Then the function* $1_C$ *defined as*

$$1_C(x) := \begin{cases} 1, & \text{if } x \in C, \\ 0, & \text{if } x \notin C, \end{cases} \tag{33.18}$$

*is Riemann integrable on* $[0,1]$ *while being discontinuous at all points of* $C$*, which is uncountable (in fact,* $C$ *is of the cardinality of the continuum).*

*Proof.* The defining expression (33.17) states that $C$ is the set of reals in $[0,1]$ whose base-3 expansion uses only 0's and 2's (and no 1's). Another way to describe $C$ is to take the unit interval $[0,1]$ and remove from it the middle open 1/3-interval, from the resulting two closed intervals remove their open middle-1/3 interval, etc. To define this formally we set $C_0 := [0,1]$ and for each natural $n \geqslant 1$,

$$C_n := \bigcup_{\sigma_0,\dots,\sigma_{n-1}\in\{0,1\}} \left( \sum_{i=0}^{n-1} \frac{2\sigma_i}{3^{i+1}} + [0, 3^{-n}] \right) \tag{33.19}$$

where we abbreviate $x + A := \{x + y : y \in A\}$. Then, being a finite union of closed intervals, $C_n$ is closed and $C_{n+1} \subseteq C_n$ holds for each $n \in \mathbb{N}$. We have $C = \bigcap_{n\in\mathbb{N}} C_n$.

To prove the desired statement, use that $C_n$ is the union of $2^n$ closed intervals of length $3^{-n}$ each separated by distance at least $3^{-n}$ from each other. Therefore, the set

$$D_n := \left\{ x \in [0,1] : \operatorname{dist}(x, C_n) \leqslant \frac{1}{4} 3^{-n} \right\}, \tag{33.20}$$

where $\operatorname{dist}(x, A) := \inf\{|x-y| : y \in A\}$, still consists of $2^n$ closed intervals of length $\frac{3}{2}3^{-n}$. Let $\Pi_n$ be the partition consisting of 0 and 1 and the endpoints of these intervals in $[0,1]$. Since the oscillation is bounded by 1 on the intervals in $D_n$ while it vanishes on the remaining intervals, we have

$$U(f, \Pi_n) - L(f, \Pi_n) \leqslant 2^n \frac{3}{2} 3^{-n} = \left(\frac{2}{3}\right)^{n-1} \tag{33.21}$$

The right-hand side tends to zero as $n \to \infty$ and so it is smaller than a given $\epsilon > 0$ once $n$ is sufficiently large. Hence, $1_C$ is Riemann integrable by Theorem 32.9.

That $C$ is uncountable follows from it being in one-to-one correspondence with $\{0,1\}^{\mathbb{N}}$, which is uncountable (in fact, equinumerous to $\mathbb{R}$ and so of cardinality of the continuum) by Cantor's diagonal argument. To see that $1_C$ is discontinuous at all points of $C$ we note that $C$ has no isolated points and that $[0,1] \smallsetminus C$ is dense in $[0,1]$. Each point of $C$ is thus a limit point of $C$ and a limit point of $[0,1] \smallsetminus C$. (We leave the proofs of these elementary facts to the reader.) □

To summarize the above observations, what mattered in all the examples above was that the function of concern was bounded and that the set of its discontinuities of size larger than any given positive number could be covered by a finite union of intervals whose total length is less than any given positive number. As we will show in the next section, this is nearly what in fact characterizes the whole class of Riemann integrable functions in full generality.