32. Darboux integral

We proceed to give another version of integration theory pioneered by G. Darboux. While this version turns out to be equivalent to Riemann's, some parts of Darboux approach become useful in their further generalization to the Lebesgue integral (which we will not discuss in this course as it requires the concept of a measure).

32.1 Darboux integrability and integral.

While Riemann's approach to integration relies on the (metric) completeness of the reals, Darboux's approach is based on the ordering property of the reals and the least upper bound axiom. We start with:

Definition 32.1 (Upper/lower Darboux sum) *Let $a < b$ be reals and $f\colon [a,b] \to \mathbb{R}$ a bounded function. Given a partition Π consisting of points $\{t_i\}_{i=1}^n$ satisfying* (31.1)*, we define the* upper Darboux sum *by*

$$U(f,\Pi) := \sum_{i=1}^n \Bigl[\sup_{x\in[t_{i-1},t_i]} f(x)\Bigr](t_i - t_{i-1}) \tag{32.1}$$

and its lower Darboux sum *by*

$$L(f,\Pi) := \sum_{i=1}^n \Bigl[\inf_{x\in[t_{i-1},t_i]} f(x)\Bigr](t_i - t_{i-1}) \tag{32.2}$$

We will write these even for Π denoting a marked partition while noting that the sums do not depend on the marked points.

The following is easy to check:

Lemma 32.2 *Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then for any marked partition Π, the associated Riemann sum $R(f,\Pi)$ from* (31.4) *obeys*

$$L(f,\Pi) \leqslant R(f,\Pi) \leqslant U(f,\Pi) \tag{32.3}$$

Moreover, for each $\epsilon > 0$ there are marked partitions Π and Π' such that

$$R(f,\Pi) \geqslant U(f,\Pi) - \epsilon \ \wedge\ R(f,\Pi') \leqslant L(f,\Pi) + \epsilon \tag{32.4}$$

The numbers $L(f,\Pi)$ and $R(f,\Pi)$ mark the interval of values that the Riemann sum can take for given partition points $\{t_i\}_{i=0}^n$. We leave the easy proof as an exercise.

Another way to think of $U(f,\Pi)$ is as the minimal area under the graphs of functions that exceed (or, more precisely, is no less than) f and are constant on each interval $(t_{i-1},t_i]$. Similarly, $L(f,\Pi)$ as then the maximal area for all such functions that are less (or, more precisely, no larger) than f.

The advantage of the Darboux sums over Riemann's is that they have several useful monotonicity properties. The highlight of these is the content of:

Lemma 32.3 *Let $f\colon [a,b] \to \mathbb{R}$ be bounded and given two partitions Π and Π' of $[a,b]$, write $\Pi \subseteq \Pi'$ if all partition points of Π occur among those of Π'. Then for all parititions Π and Π',*

$$\Pi \subseteq \Pi' \ \Rightarrow\ L(f,\Pi) \leqslant L(f,\Pi') \leqslant U(f,\Pi') \leqslant U(f,\Pi) \tag{32.5}$$

In particular, we have

$$\forall \Pi, \Pi' \text{ partitions of } [a,b]: \quad L(f,\Pi) \leqslant U(f,\Pi') \tag{32.6}$$

Proof. We start with (32.5). Here it suffices to assume that Π' consists of Π and one point u that lies in interval $[t_{i-1}, t_i]$. In this case

$$\begin{aligned} U(f,\Pi') - U(f,\Pi) = \Big[\sup_{x\in[t_{i-1},u]} f(x)\Big](u-t_{i-1}) \\ + \Big[\sup_{x\in[u,t_i]} f(x)\Big](t_i-u) - \Big[\sup_{x\in[t_{i-1},t_i]} f(x)\Big](t_i-t_{i-1}) \end{aligned} \tag{32.7}$$

Using that

$$\max\Big\{\sup_{x\in[t_{i-1},u]} f(x), \sup_{x\in[t_{i-1},u]} f(x)\Big\} \leqslant \sup_{x\in[t_{i-1},t_i]} f(x) \tag{32.8}$$

we then get $U(f,\Pi') - U(f,\Pi) \leqslant 0$ as desired. The inequality $L(f,\Pi) \leqslant L(f,\Pi')$ is proved similarly.

In order to get (32.6) from this, let Π and Π' be arbitrary partitions and let $\Pi \cup \Pi'$ be a partition consisting of all the points in both Π and Π'. Then $\Pi \cup \Pi' \subseteq \Pi$ and $\Pi \cup \Pi' \subseteq \Pi'$ and so

$$L(f,\Pi) \leqslant L(f,\Pi\cup\Pi') \leqslant U(f,\Pi\cup\Pi') \leqslant U(f,\Pi') \tag{32.9}$$

holds by (32.5). □

We now put forward:

Definition 32.4 (Upper/lower Darboux integral) *Let $f: [a,b] \to \mathbb{R}$ be bounded. Then*

$$\overline{\int_a^b} f(x)\mathrm{d}x := \inf\big\{U(f,\Pi) \colon \Pi \text{ is a partition of [a,b]}\big\} \tag{32.10}$$

is the upper Darboux integral *while*

$$\underline{\int_a^b} f(x)\mathrm{d}x := \sup\big\{L(f,\Pi) \colon \Pi \text{ is a partition of [a,b]}\big\} \tag{32.11}$$

is the lower Darboux integral.

Since $U(f,\Pi)$ is defined using suprema, the upper Darboux integral can be thought of as a version of "limsup" and, similarly, the lower Darboux integral as a version of "liminf." As a consequence of (32.6), we get:

Corollary 32.5 *For all bounded $f: [a,b] \to \mathbb{R}$ we then have*

$$\underline{\int_a^b} f(x)\mathrm{d}x \leqslant \overline{\int_a^b} f(x)\mathrm{d}x \tag{32.12}$$

We can think of this as the analogy of statement that "liminf" is no larger than the "limsup." The existence of the "limit," which is what we have (albeit in a different form) associated with Riemann integrability, is then characterized by the "limsup" being equal to the "liminf:"

Definition 32.6 (Darboux integrability/integral) *We say that a bounded* $f\colon [a,b] \to \mathbb{R}$ *is* Darboux integrable (DI) *if*

$$\underline{\int_a^b} f(x)\mathrm{d}x = \overline{\int_a^b} f(x)\mathrm{d}x \tag{32.13}$$

The common value of these quantities is then called the Darboux integral.

32.2 Equivalence of Darboux and Riemann integral.

As announced earlier, the two approaches to integral turn out to be equivalent. We state this along with a convenient necessary and sufficient condition:

Lemma 32.7 (Characterization of Darboux integrability) *Let* $f\colon [a,b] \to \mathbb{R}$*. Then*

$$f \text{ DI} \;\Leftrightarrow\; \forall\epsilon > 0\, \exists\Pi = \text{partition of } [a,b]\colon\; U(f,\Pi) - L(f,\Pi) < \epsilon \tag{32.14}$$

Proof. For any partition Π we have

$$\overline{\int_a^b} f(x)\mathrm{d}x - \underline{\int_a^b} f(x)\mathrm{d}x \leqslant U(f,\Pi) - L(f,\Pi) \tag{32.15}$$

With the help of (32.12), this proves $\Leftarrow$ in (32.14).

On the other hand, by the properties of suprema and infima, for each $\epsilon > 0$ there are partitions Π and Π' such that

$$U(f,\Pi) \leqslant \overline{\int_a^b} f(x)\mathrm{d}x + \epsilon \quad\wedge\quad L(f,\Pi') \geqslant \underline{\int_a^b} f(x)\mathrm{d}x - \epsilon. \tag{32.16}$$

In light of (32.9), the common refinement $\Pi'' := \Pi \cup \Pi'$ of Π and Π' obeys

$$\begin{aligned} U(f,\Pi'') - L(f,\Pi'') &\leqslant U(f,\Pi) - L(f,\Pi') \\ &\leqslant \overline{\int_a^b} f(x)\mathrm{d}x - \underline{\int_a^b} f(x)\mathrm{d}x + 2\epsilon = 2\epsilon, \end{aligned} \tag{32.17}$$

where the last equality follow from (32.13) and the assumption that f is Darboux integrable. This proves $\Rightarrow$ in (32.14). ☐

Note that the previous proof actually shows:

Corollary 32.8 *For all* $f\colon [a,b] \to \mathbb{R}$ *bounded,*

$$\overline{\int_a^b} f(x)\mathrm{d}x - \underline{\int_a^b} f(x)\mathrm{d}x = \inf\Big\{U(f,\Pi) - L(f,\Pi)\colon \Pi = \text{partition of } [a,b]\Big\} \tag{32.18}$$

An important point in applications is that the quantity $U(f,\Pi) - L(f,\Pi)$ can be expressed using the concept of (unrestricted) *oscillation*, which for a real-valued function f on a non-empty set $A \subseteq \mathrm{Dom}(f)$ is defined as

$$\mathrm{osc}(f,A) := \sup\big\{|f(y) - f(x)|\colon x,y \in A\big\} \tag{32.19}$$

(this corresponds to $\operatorname{osc}_f(f,\infty)$ in our earlier notation). To see why this is relevant for us, note that

$$\operatorname{osc}(f,A)=\sup_{x\in A}f(x)-\inf_{x\in A}f(x) \tag{32.20}$$

and so for any $f\colon[a,b]\to\infty$ and any partition $\Pi=\{t_i\}_{i=0}^n$ of $[a,b]$,

$$U(f,\Pi)-L(f,\Pi)=\sum_{i=1}^n\operatorname{osc}\bigl(f,[t_{i-1},t_i]\bigr)(t_i-t_{i-1}). \tag{32.21}$$

The integrability thus (vaguely) corresponds to there being a sequence of partitions along which the total length of intervals where the oscillation is appreciable can be made arbitrarily small. Note that, unlike for the Riemann integral, we make no restrictions as to the mesh of the partition used but, in all but the trivial cases, the partitions Π with small value of $U(f,\Pi)-L(f,\Pi)$ will have small mesh.

The main result of this section is:

Theorem 32.9 (Equivalence of Darboux and Riemann) *For all $f\colon[a,b]\to\mathbb{R}$ be bounded:*
(1) *Riemann integrability of f on $[a,b]$ implies*

$$\forall\epsilon>0\,\exists\Pi=\text{ partition of }[a,b]\colon\quad U(f,\Pi)-L(f,\Pi)<\epsilon \tag{32.22}$$

(2) *Condition* (32.22) *implies Riemann integrability of f on $[a,b]$.*
In particular, for all bounded or all $f\colon[a,b]\to\mathbb{R}$,

$$f\text{ RI}\ \Leftrightarrow\ f\text{ DI} \tag{32.23}$$

and, if these are TRUE, then the Darboux and the Riemann integrals coincide.

Proof. Let us start with (1). Fix $\epsilon>0$. By the arguments underlying the proof of Lemma 32.2, for any choices of partition points, there are choices of marked points such that the associated partitions Π and Π' obey

$$R(f,\Pi)\geqslant U(f,\Pi)-\epsilon\ \wedge\ R(f,\Pi')\leqslant L(f,\Pi')+\epsilon \tag{32.24}$$

Since f is assumed Riemann integrable, the Cauchy criterion for Riemann integrability (Lemma 31.7) implies that there is $\delta>0$ such that for any marked partitions $\|\Pi\|<\delta$ and $\|\Pi'\|<\delta$ then

$$\bigl|R(f,\Pi)-R(f,\Pi')\bigr|<\epsilon \tag{32.25}$$

Picking the partition points so that $\|\Pi\|<\delta$ and $\|\Pi'\|<\delta$ holds and the marked points so that (32.24) is TRUE, we then get

$$\begin{aligned}U(f,\Pi\cup\Pi')-L(f,\Pi\cup\Pi')&\leqslant U(f,\Pi)-L(f,\Pi')\\&\leqslant\bigl|R(f,\Pi)-R(f,\Pi')\bigr|+2\epsilon<3\epsilon\end{aligned} \tag{32.26}$$

where $\Pi\cup\Pi'$ is the common refinement of Π and Π' and where the first inequality follows from (32.9). The claim follows from (32.21) by relabeling 3ϵ as ϵ.

For (2) we pick $\epsilon>0$ and let $\Pi_0=\{t_i\}_{i=0}^n$ be a partition of $[a,b]$ such that

$$U(f,\Pi_0)-L(f,\Pi_0)<\epsilon. \tag{32.27}$$

Pick $\delta>0$ satisfying

$$3n\delta\Bigl(\sup_{x\in[a,b]}\bigl|f(x)\bigr|\Bigr)<\epsilon \tag{32.28}$$

Given marked partitions Π and Π' such that $\|\Pi\| < \delta$ and $\|\Pi'\| < \delta$, let $\Pi_0 \cup \Pi'$, resp., $\Pi_0 \cup \Pi''$ be marked partitions with the partition points of both Π_0 and Π, resp., both Π_0 and Π' and the marked points of Π', resp., Π'' in the intervals of $\Pi_0 \cup \Pi'$, resp., $\Pi_0 \cup \Pi''$ where the marked points of these partition fall into and left endpoints of the intervals in those where they do not. The argument underlying (31.33) then shows

$$\big|R(f,\Pi_0 \cup \Pi) - R(f,\Pi)\big| + \big|R(f,\Pi_0 \cup \Pi') - R(f,\Pi')\big| < 2\epsilon. \tag{32.29}$$

But (32.3) and (32.5) give

$$L(f,\Pi_0) \leqslant L(f,\Pi_0 \cup \Pi) \leqslant R(f,\Pi_0 \cup \Pi) \leqslant U(f,\Pi_0 \cup \Pi) \leqslant U(f,\Pi_0) \tag{32.30}$$

and similarly for $R(f,\Pi_0 \cup \Pi')$. Therefore,

$$\begin{aligned} \big|R(f,\Pi) - R(f,\Pi')\big| &\leqslant 2\epsilon + \Big|R(f,\Pi_0 \cup \Pi) - R(f,\Pi_0 \cup \Pi')\Big| \\ &\leqslant 2\epsilon + U(f,\Pi_0) - L(f,\Pi_0) < 3\epsilon \end{aligned} \tag{32.31}$$

We have thus verified the Cauchy criterion from Lemma 31.7 and proved that f is Riemann integrable. With f both Riemann and Darboux integrable, the integrals coincide thanks to (32.3) and the definition of the upper and lower Darboux integral. □

Since the two concepts of integrability are thus shown to be equivalent, we will henceforth refer to both Darboux's and Riemann's integral simply as the Riemann integral. Darboux's integral is often simpler to work with conceptually as it easier to verify its main criterion for integrability (32.22). (Recall that the harder part of the proof of Theorem 32.9 was the implication "(32.22) ⇒ Riemann integrability.") This gives us the following simple corollary:

Corollary 32.10 *Let $f,g\colon [a,b] \to \mathbb{R}$ be bounded. Then*

(1) *f Riemann integrable ⇒ $|f|$ Riemann integrable*
(2) *f,g Riemann integrable ⇒ $f \cdot g$ Riemann integrable*

Proof. This follows from the easy bounds

$$\operatorname{osc}\big(|f|,A\big) \leqslant \operatorname{osc}\big(f,A\big) \tag{32.32}$$

and

$$\operatorname{osc}\big(f\cdot g,A\big) \leqslant \|f\|\operatorname{osc}\big(g,A\big) + \|g\|\operatorname{osc}\big(f,A\big) \tag{32.33}$$

We leave the remaining details to the reader. □

On the other hand, some statements are slightly harder to show in Darboux's approach than in Riemann's approach. One of these is additivity with respect to the integrand which in the context of Darboux integral takes the following form:

Lemma 32.11 *Let $a < b$ be reals and $f,g\colon [a,b] \to \mathbb{R}$ bounded functions. Then the upper Darboux integral is subadditivive,*

$$\overline{\int_a^b} (f+g)(x)\,\mathrm{d}x \leqslant \overline{\int_a^b} f(x)\,\mathrm{d}x + \overline{\int_a^b} g(x)\,\mathrm{d}x \tag{32.34}$$

and the lower Darboux integral is superadditive,

$$\underline{\int_a^b}(f+g)(x)\mathrm{d}x \geqslant \underline{\int_a^b} f(x)\mathrm{d}x + \underline{\int_a^b} g(x)\mathrm{d}x \tag{32.35}$$

In particular, if both f and g are Darboux integrable, then so is $f+g$ and the Darboux integral of $f+g$ is the sum of Darboux integrals of f and g.

We leave the proof of this lemma to homework while noting that one can also construct non-Darboux integrable f and/or g such that the inequalities in (32.34–32.35) are strict. The upper/lower Darboux integrals are also positive homogeneous, meaning that the upper integral of λf is λ-multiple of the upper integral of f for all $\lambda \geqslant 0$. On the other hand, multiplying f by negative λ transforms the upper integral of λf to the λ-multiple of the lower integral and *vice versa*.

The main disadvantage of Darboux's approach is that is based on the ordering property of $\mathbb{R}$, which is unavailable as soon as the integrand takes value in more complicated sets (such as the generalization of the Riemann integral to vector-valued functions) or we need to work with the generalization of the Riemann integral to what we will later call the Stieltjes integral.