31. Riemann integral

An important early achievement of the rigorous approach to mathematics in mid 19th century was the definition of (what we now call) the Riemann integral. We will follow the original ideas of Riemann while the textbook focuses on the approach due to Darboux which, as we will also show, is equivalent to Riemann's. Both approaches have their own merit on their own as well as in various extensions.

31.1 Area under a curve.

The concept of an integral was first introduced by Newton and Leibnitz in their treatments of differential and integral calculus. The idea was to express the area under the graph of a function by adding areas of infinitesimal rectangles. Riemann formalized this precisely using the following concepts:

Definition 31.1 (Marked partition and Riemann sum) *Let $a < b$ be reals. A* marked partition Π *of interval $[a,b]$ is a pair of sequences $\{t_i\}_{i=0}^n$ and $\{t_i^\star\}_{i=1}^n$ such that*

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b \tag{31.1}$$

and

$$\forall i = 1,\dots,n\colon \quad t_i^\star \in [t_{i-1}, t_i] \tag{31.2}$$

The mesh *of Π is then defined as*

$$\|\Pi\| := \max_{i=1,\dots,n} |t_i - t_{i-1}| \tag{31.3}$$

Given a function $f\colon [a,b] \to \mathbb{R}$,

$$R(f,\Pi) := \sum_{i=1}^n f(t_i^\star)(t_i - t_{i-1}) \tag{31.4}$$

is the Riemann sum *associated with marked partition Π of interval $[a,b]$.*

The quantity $f(t_i^\star)(t_i - t_{i-1})$ represents the area of a rectangle with base $[t_{i-1}, t_i]$ and height $f(t_i^\star)$. (This really applies only if $f(t_i^\star) > 0$; if this value is negative, we get the negative area.) The Riemann sum $R(f,\Pi)$ is then the aggregate area of these rectangles which we can then also think of as the area under the piece-wise constant curve that has height $f(t_i^\star)$ above $(t_{i-1}, t_i]$. We then put forward:

Definition 31.2 (Riemann integrability) *We say that $f\colon [a,b] \to \mathbb{R}$ is* Riemann integrable *if there exists $L \in \mathbb{R}$ such that for each $\epsilon > 0$ there is $\delta > 0$ such that for all marked partitions Π of $[a,b]$,*

$$\|\Pi\| < \delta \;\Rightarrow\; \big|R(f,\Pi) - L\big| < \epsilon \tag{31.5}$$

The logical proposition "$\forall \epsilon > 0\, \exists \delta > 0\, \forall \Pi =$ marked partition: (31.5) holds" will at times be abbreviated as

$$\lim_{\|\Pi\| \to 0} R(f,\Pi) = L \tag{31.6}$$

while noting that this is not a limit in the sense used earlier (although it can be phrased as convergence of nets). Notwithstanding, exactly the same argument as for limits can be used to show that L above is unique, if it exists:

Lemma 31.3 *If the property in Definition 31.2 holds for L and L', then $L = L'$.*

Proof. Fix $\epsilon > 0$ and let $\delta > 0$ be such that $\|\Pi\| < \delta$ implies $|R(f,\Pi) - L| < \epsilon$. Similarly, let $\delta' > 0$ be such that $\|\Pi\| < \delta'$ implies $|R(f,\Pi) - L| < \epsilon$. Since a marked partition Π exists such that both $\|\Pi\| < \delta$ and $\|\Pi\| < \delta'$, we thus have

$$|L - L'| \leqslant \bigl|R(f,\Pi) - L\bigr| + \bigl|R(f,\Pi) - L'\bigr| < 2\epsilon. \tag{31.7}$$

As this holds for all $\epsilon > 0$, we have $L = L'$. □

The uniqueness justifies introduction of a special symbol:

Definition 31.4 (Riemann integral) *Given reals $a < b$ and a function $f\colon [a,b] \to \mathbb{R}$, the* Riemann integral *of f on interval $[a,b]$ is defined as*

$$\int_a^b f(x)\mathrm{d}x := \lim_{\|\Pi\|\to 0} R(f,\Pi) \tag{31.8}$$

whenever f is Riemann integrable on $[a,b]$.

We will write "f RI" to denote the phrase "f is Riemann integrable" whenever condensed notation is desired. The convention

$$b < a \;\Rightarrow\; \int_b^a f(t)\mathrm{d}t := -\int_a^b f(t)\mathrm{d}t \tag{31.9}$$

is used for convenience (and because it works nicely in manipulations that we will consider later and fits a corresponding property of the Stieltjes integral).

We note that the above concepts were known already to Newton and Leibnitz who also understood that one has to take the mesh of Π to zero in order to approximate the area under the graph of f better and better. When the notion of a limit became understood precisely, attempts were even made (for instance, by Cauchy who worked with continuous functions) to come up with conditions on f that would guarantee that such approximations converged. Riemann's approach is qualitatively different in that, instead of trying to establish the convergence under increasingly relaxed conditions on the regularity of f, he made integrability a regularity property in its own right.

31.2 Linearity and additivity.

It is easy to check that constant functions are Riemann integrable so the above theory is not vacuous. In order to demonstrate the use of Riemann integrability properly, let us prove some basic properties of the Riemann integral. We start with:

Lemma 31.5 (Linearity) *Let $a < b$ be reals and assume that $f,g\colon [a,b] \to \mathbb{R}$ are Riemann integrable on $[a,b]$. Then for all $\alpha,\beta \in \mathbb{R}$, also $\alpha f + \beta g$ is Riemann integrable on $[a,b]$ and*

$$\int_a^b \bigl(\alpha f(x) + \beta g(x)\bigr)\mathrm{d}x = \alpha\int_a^b f(x)\mathrm{d}x + \beta\int_a^b g(x)\mathrm{d}x \tag{31.10}$$

Proof. Let $\alpha,\beta \in \mathbb{R}$ be fixed and f,g be RI on $[a,b]$. Fix $\epsilon > 0$. Then the RI of f ensures existence of $\delta > 0$ be such that for all marked partitions Π of $[a,b]$,

$$\|\Pi\| < \delta \;\Rightarrow\; \left|R(f,\Pi) - \int_a^b f(x)\mathrm{d}x\right| < \frac{\epsilon}{1 + |\alpha| + |\beta|} \tag{31.11}$$

and let $\delta' > 0$ be such that for all marked partitions Π of $[a,b]$,

$$\|\Pi\| < \delta' \;\Rightarrow\; \left|R(g,\Pi) - \int_a^b g(x)\mathrm{d}x\right| < \frac{\epsilon}{1+|\alpha|+|\beta|} \tag{31.12}$$

Since the definition (31.4) implies

$$R\bigl(\alpha f + \beta g, \Pi\bigr) = \alpha R(f,\Pi) + \beta R(g,\Pi) \tag{31.13}$$

whenever Π is such that $\|\Pi\| < \min\{\delta,\delta'\}$, the triangle inequality shows

$$\begin{aligned}
&\left|R\bigl(\alpha f + \beta g,\Pi\bigr) - \alpha\int_a^b f(x)\mathrm{d}x - \beta\int_a^b g(x)\mathrm{d}x\right| \\
&\qquad = \left|\alpha\Bigl[R(f,\Pi) - \int_a^b f(x)\mathrm{d}x\Bigr] + \beta\Bigl[R(g,\Pi) - \int_a^b g(x)\mathrm{d}x\Bigr]\right| \\
&\qquad \leqslant |\alpha|\left|R(f,\Pi) - \int_a^b f(x)\mathrm{d}x\right| + |\beta|\left|R(g,\Pi) - \int_a^b g(x)\mathrm{d}x\right| \\
&\qquad < \bigl(|\alpha|+|\beta|\bigr)\frac{\epsilon}{1+|\alpha|+|\beta|} < \epsilon
\end{aligned} \tag{31.14}$$

As this holds for all $\epsilon > 0$, we have proved that $\alpha f + \beta g$ is RI and

$$\lim_{\|\Pi\|\to 0} R\bigl(\alpha f + \beta g, \Pi\bigr) = \alpha\int_a^b f(x)\mathrm{d}x + \beta\int_a^b g(x)\mathrm{d}x \tag{31.15}$$

This is the desired claim. □

Another well-known property of the integral concerns the its additivity under subdivision of $[a,b]$. This is the content of:

Lemma 31.6 (Additivity) *Let $a < c < b$ be reals and let $f\colon [a,b] \to \mathbb{R}$. Then*

$$f \text{ RI on } [a,b] \;\Leftrightarrow\; f \text{ RI on } [a,c] \,\wedge\, f \text{ RI on } [c,b] \tag{31.16}$$

and when both (equivalent) statements are TRUE, then

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x \tag{31.17}$$

Our strategy of the proof is to first show ⇒ in (31.16) and then prove ⇐ along with the formula (31.17). The former requires:

Lemma 31.7 (Cauchy criterion for RI) *Let $f\colon [a,b] \to \mathbb{R}$. Then*

$$f \text{ RI} \;\Leftrightarrow\; \inf_{\delta>0}\sup_{\|\Pi\|,\|\Pi'\|<\delta}\bigl|R(f,\Pi) - R(f,\Pi')\bigr| = 0 \tag{31.18}$$

where Π and Π' on the right denote marked partitions of $[a,b]$.

Leaving the easy proof to homework, we now show:

Proof of ⇒ in (31.16). Assume f is RI on $[a,b]$. Pick $\epsilon > 0$ and let $\delta > 0$ be such that the supremum in (31.18) is less than ϵ. Now pick any marked two partitions Π_1 and Π_1' of $[a,c]$ and any marked partitions Π_2 and Π_2' of $[c,b]$, all with mesh less than δ. Relabeling if necessary we may assume that

$$R(f,\Pi_1) \leqslant R(f,\Pi_1') \,\wedge\, R(f,\Pi_2) \leqslant R(f,\Pi_2') \tag{31.19}$$

Now let Π be the marked partition of $[a,b]$ obtained by concatenating Π_1 and Π_2 and let Π' be the marked partition of $[a,b]$ obtained by concatenating Π_1 and Π_2'. (Technically, if $\Pi_1 = (\{u_i\}_{i=0}^n, \{u_i^\star\}_{i=1}^n)$ and $\Pi_1 = (\{v_i\}_{i=0}^m, \{v_i^\star\}_{i=0}^m)$ then Π is the partition $\Pi = (\{t_i\}_{i=0}^{n+m}, \{t_i^\star\}_{i=0}^{n+m})$ where $t_i := u_i$ and $t_i^\star := u_i^\star$ for $i \leqslant n$ and $t_{n+j} := v_j$ and $t_{n+j}^\star := v_j^\star$ for $j \leqslant m$.) Then

$$R(f,\Pi) = R(f,\Pi_1) + R(f,\Pi_2) \ \wedge\ R(f,\Pi') = R(f,\Pi_1') + R(f,\Pi_2') \tag{31.20}$$

and so

$$0 \leqslant R(f,\Pi_1') - R(f,\Pi_1) + R(f,\Pi_2') - R(f,\Pi_2) = R(f,\Pi') - R(f,\Pi) < \epsilon \tag{31.21}$$

where the last bound follow from the fact that $\|\Pi\| = \max\{\|\Pi_1\|, \|\Pi_2\|\} < \delta$ and similarly $\|\Pi'\| = \max\{\|\Pi_1'\|, \|\Pi_2'\|\} < \delta$.

In light of (31.20), hereby we concluded that, once the marked partitions Π_1 and Π_1' of $[a,c]$ and marked partitions Π_2 and Π_2' of $[c,b]$ have mesh less than δ, then

$$\Big|R(f,\Pi_1') - R(f,\Pi_1)\Big| < \epsilon \ \wedge\ R(f,\Pi_2') - R(f,\Pi_2) < \epsilon \tag{31.22}$$

Thanks to (31.18), this proves that f is RI on both $[a,c]$ and $[c,b]$ as claimed. □

In order to address the reverse implication in (31.16), we need:

Lemma 31.8 *Let $f\colon [a,b] \to \mathbb{R}$. Then*

$$f \text{ RI on } [a,b] \ \Rightarrow\ f \text{ bounded on } [a,b] \tag{31.23}$$

In particular, if f isRiemann integrable on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leqslant \Big(\sup_{x\in[a,b]}\big|f(x)\big|\Big)\,|b-a| \tag{31.24}$$

Proof. Suppose f is Riemann integrable. Then there is $\delta > 0$ such that for any marked partition Π with $\|\Pi\| < \delta$, the triangle inequality shows

$$\big|R(f,\Pi)\big| \leqslant \left|\int_a^b f(x)\mathrm{d}x\right| + 1 \tag{31.25}$$

Writing $\{t_i\}_{i=0}^n$ for the points of the partition and $\{t_i^\star\}_{i=1}^n$ for the marked points, another use of the triangle inequality gives for any $i = 1,\dots,n$,

$$\big|f(t_i^\star)\big|\,|t_i - t_{i-1}|\big| \leqslant \sum_{j\neq i}\big|f(t_j^\star)\big||t_j - t_{j-1}| + \left|\int_a^b f(x)\mathrm{d}x\right| + 1. \tag{31.26}$$

Optimizing over $t_i^\star$, we thus get

$$\sup_{t\in[t_{i-1},t_i]}\big|f(t)\big| \leqslant \frac{1}{|t_i - t_{i-1}|}\text{ RHS of (31.26)} \tag{31.27}$$

As this holds for all $i = 1,\dots,n$, the function f is bounded as claimed.

As to (31.24), observe that

$$\begin{aligned}\bigl|R(f,\Pi)\bigr| &\leqslant \sum_{i=1}^{n}\bigl|f(t_i^\star)\bigr|(t_i-t_{i-1})\\ &\leqslant \Bigl(\sup_{x\in[a,b]}\bigl|f(x)\bigr|\Bigr)\sum_{i=1}^{n}(t_i-t_{i-1}) = \Bigl(\sup_{x\in[a,b]}\bigl|f(x)\bigr|\Bigr)(b-a)\end{aligned} \tag{31.28}$$

Using (31.8), this now extends to the Riemann integral. □

We are now ready to give:

Proof of Lemma 31.6. The direction $\Rightarrow$ in (31.16) has been proved above, so we only need to show $\Rightarrow$ and prove the formula (31.17). Assume f to be RI on $[a,c]$ and $[c,b]$ and fix $\epsilon > 0$. Then there exist $\delta', \delta'' > 0$ such that if Π' is a partition of $[a,c]$ with $\|\Pi'\| < \delta'$ and Π'' is a partition of $[c,b]$ with $\|\Pi''\| < \delta''$ then

$$\left|R(f,\Pi') - \int_a^c f(x)\mathrm{d}x\right| < \epsilon \ \wedge\ \left|R(f,\Pi'') - \int_c^b f(x)\mathrm{d}x\right| < \epsilon \tag{31.29}$$

(The intervals the Riemann sums are over are clear from the partition.)

Let now Π be a marked partition of $[a,b]$ with $\|\Pi\| < \min\{\delta',\delta''\}$. If Π contains c (in the sequence defining partition points), then Π splits into two marked partitions, Π' and Π'' of $[a,c]$ and $[c,b]$, respectively, and we have

$$R(f,\Pi) = R(f,\Pi') + R(f,\Pi''). \tag{31.30}$$

It follows that

$$\begin{aligned}&\left|R(f,\Pi) - \int_a^c f(x)\mathrm{d}x - \int_c^b f(x)\mathrm{d}x\right|\\ &\qquad\leqslant \left|R(f,\Pi') - \int_a^c f(x)\mathrm{d}x\right| + \left|R(f,\Pi'') - \int_c^b f(x)\mathrm{d}x\right| < 2\epsilon\end{aligned} \tag{31.31}$$

The problem is that this bound is restricted to partitions containing c.

Consider now a marked partition $\widetilde{\Pi}$ of $[a,b]$ that does NOT contain c and let $[t_{i-1},t_i]$ be the unique interval in this partition such that $c \in (t_{i-1},t_i)$. Let Π be the partition obtained by adding c to $\widetilde{\Pi}$ and the marked points $u^\star \in [t_{i-1},c]$ and $v^\star \in [c,t_i]$. Then

$$R(f,\Pi) - R(f,\widetilde{\Pi}) = f(t_i^\star)(t_i-t_{i-1}) - f(u^\star)(c-t_{i-1}) - f(v^\star)(t_i-c) \tag{31.32}$$

and, since by Lemma 31.8 f is bounded on $[a,c]$ and $[c,b]$ and thus on $[a,b]$,

$$\bigl|R(f,\Pi) - R(f,\widetilde{\Pi})\bigr| \leqslant 3\Bigl(\sup_{x\in[a,b]}\bigl|f(x)\bigr|\Bigr)\,\|\Pi\|. \tag{31.33}$$

Let δ be such that

$$0 < \delta < \min\{\delta',\delta''\} \ \wedge\ 3\delta\Bigl(\sup_{x\in[a,b]}\bigl|f(x)\bigr|\Bigr) < \epsilon \tag{31.34}$$

If $\|\widetilde{\Pi}\| < \delta$, then (31.31) and (31.33) (along with the fact that $\|\Pi\| \leqslant \|\widetilde{\Pi}\|$) show

$$\left|R(f,\widetilde{\Pi}) - \int_a^c f(x)\mathrm{d}x - \int_c^b f(x)\mathrm{d}x\right| \leqslant \left|R(f,\Pi) - R(f,\widetilde{\Pi})\right| + \left|R(f,\Pi) - \int_a^c f(x)\mathrm{d}x - \int_c^b f(x)\mathrm{d}x\right| \leqslant 3\epsilon \tag{31.35}$$

This proves the implication $\Rightarrow$ in (31.16) and shows (31.17) as well. □

We note that the fact that Riemann integrability requires boundedness is actually the first sign of problems with the whole concept. Indeed, the function $f(x) := \frac{1}{\sqrt{x}}$ is not Riemann integrable on $[0,1]$ while (as we will show later) it is Riemann integrable on $[a,1]$ for any $a \in (0,1)$ with a well defined limit as $a \to 0^+$. This function can still be included into the theory by using the notion of an *improper integral* but that then unfortunately fails other important properties that the “proper” Riemann integral has.