28. Discontinuities and functions of bounded variation

We will now take a closer look at functions at their points of discontinuity. Attention will particularly be paid to monotone functions and then to functions of bounded variation.

28.1 Discontinuities of first and second kind.

We start by a concept that exist in full generality of functions on metric spaces:

Definition 28.1 *Let* $f\colon X\to Y$ *be a function between metric spaces* X *and* Y *and* $x\in X$ *a limit point of* $\text{Dom}(f)$*. We say that* f *has a* removable singularity *at* x *if*

$$\lim_{z\to x} f(z)\text{ exists}\ \wedge\ \Bigl(x\in\text{Dom}(f)\ \Rightarrow\ \lim_{z\to x} f(z)\neq f(x)\Bigr) \tag{28.1}$$

The motivation for the name comes from Lemma 27.4 which shows that if f has a removable singularity at x then a simple re-definition of f (if $f(x)$ is already defined; otherwise this is just an extension of f) leads to a continuous function.

The remaining discussion will be restricted to functions $f\colon\mathbb{R}\to\mathbb{R}$. As should be clear from above, having the one sided limits at x is the next best thing one can hope to have if the full limit does not exist and/or the function is not continuous. This motivates:

Definition 28.2 (Discontinuities of 1st and 2nd kind) *Let* $f\colon\mathbb{R}\to\mathbb{R}$ *and* $x\in\text{int}(\text{Dom}(f))$*. We say that* f *has a* discontinuity of the first kind *at* x *if*

$$f(x^+), f(x^-)\text{ exist}\ \wedge\ \bigl|\{f(x^-), f(x^+), f(x)\}\bigr|>1 \tag{28.2}$$

We say that f *has* discontinuity of the second kind *at* x *if at least one of the limits* $f(x^+)$ *and* $f(x^-)$ *does NOT exist.*

To demonstrate these, we note:

Lemma 28.3 *Let* $f\colon\mathbb{R}\to\mathbb{R}$ *(with* $\text{Dom}(f)=\mathbb{R}$*) be monotone. Then* $f(x^+)$ *and* $f(x^-)$ *exist at all* $x\in\mathbb{R}$*. In particular,* f *has no discontinuities of the second kind.*

Proof. Suppose that f is non-decreasing (otherwise take $-f$ instead of f) and let $x\in\mathbb{R}$. We claim that $f(x^+)$ exists and, in fact,

$$f(x^+)=\inf\bigl\{f(z)\colon z>x\bigr\} \tag{28.3}$$

Indeed, since f is non-decrasing, $f(x)$ is a lower bound on every value in the set and so the infimum exists proper in $\mathbb{R}$. Writing c for the infimum, it follows that, for each $\epsilon>0$ there is z_0 such that

$$c\leqslant f(z_0)<c+\epsilon \tag{28.4}$$

Denoting $\delta:=z_0-x$, the monotonicity of f then ensures that $c\leqslant f(z)<c+\epsilon$ holds for all $z\in(x,x+\delta)$, i.e.,

$$f\bigl((x,x+\delta)\bigr)\subseteq(c-\epsilon,c+\epsilon) \tag{28.5}$$

As this applies for all $\epsilon>0$, the definition (27.26) gives $f(x^+)=c$ as desired. The left limit is treated analogously (or turned into the above by considering $x\mapsto f(-x)$). □

We note that the above arguments can be bolstered to give us even the following:

Lemma 28.4 *Let $f\colon \mathbb{R}\to\mathbb{R}$ be non-decreasing with* $\text{Dom}(f)=\mathbb{R}$. *Then $x\mapsto f(x^+)$ is right continuous while $x\mapsto f(x^-)$ is left continuous with both non-decreasing. Moreover,*

$$\forall x\in\mathbb{R}\colon\;\; f(x^-)\leqslant f(x)\leqslant f(x^+) \tag{28.6}$$

while

$$\forall x,y\in\mathbb{R}\colon\;\; x<y\;\Rightarrow\; f(x^+)\leqslant f(y^-) \tag{28.7}$$

We leave a proof of this lemma to a homework exercise. In order to give an example of a function with discontinuities of the second kind, consider the Dirichlet function $1_{\mathbb{Q}}$ defined in (27.11) that fails to have one-sided limits at every point of $\mathbb{R}$. A slightly more subtle example is the function h from (26.14) extended by zero to $(-\infty,0]$ that has discontinuity of the second kind at 0.

Proceeding in the discussion of monotone functions, we now observe:

Lemma 28.5 *Let $f\colon \mathbb{R}\to\mathbb{R}$ be monotone with* $\text{Dom}(f)=\mathbb{R}$. *Then*

$$\bigl\{x\in\mathbb{R}\colon f(x^+)\neq f(x^-)\bigr\}\text{ is finite or countable} \tag{28.8}$$

Proof. Assume without loss of generality that f is non-decreasing (otherwise, replace f by $-f$). Given a natural $m>0$ and real $\epsilon>0$, let

$$A'_{m,\epsilon}:=\bigl\{x\in(-m,m)\colon f(x^+)>f(x)+\epsilon\bigr\} \tag{28.9}$$

We claim that $A'_{m,\epsilon}$ is finite which we will show by proving a quantitative upper bound on its cardinality. Let $n\geqslant 1$ be a natural such that

$$n\epsilon>f(m^+)-f(-m^+) \tag{28.10}$$

Define $y_0:=f(-m^+)$ and, for $k=1,\dots,n$, let

$$z_k:=z_0+k\epsilon^{-1} \tag{28.11}$$

Then

$$\bigl[f(-m^+),f(m^+)\bigr]=\bigcup_{k=1}^n[z_{k-1},z_k] \tag{28.12}$$

Using that fact that $x\in(-m,m)$ implies $f(-m^+)\leqslant f(x^-)\leqslant f(x)\leqslant f(x^+)\leqslant f(m^+)$ thanks to the inequalities from Lemma 28.4, we also have

$$f(A'_{m,\epsilon})\subseteq\bigl[f(-m^+),f(m^+)\bigr] \tag{28.13}$$

and so each $z\in f(A_{m,\epsilon})$ lies in one of the intervals $\{[z_{k-1},z_k]\colon i=1,\dots,n\}$. This allows us to define $h\colon A_{m,\epsilon}\to\{1,\dots,n\}$ by

$$h(x):=\inf\bigl\{k=1,\dots,n\colon f(x)\in[z_{k-1},z_k]\bigr\} \tag{28.14}$$

Now consider $x,y\in A'_{m,\epsilon}$ with $x<y$. Then the inequalities from Lemma 28.4 give

$$\epsilon<f(x^+)-f(x)\leqslant f(y)-f(x) \tag{28.15}$$

which means that x and y cannot belong to the same interval $[z_{k-1},z_k]$. The map h is thus an injection and so $|A_{m,\epsilon}|\leqslant n$ by Lemma 11.2 from 131AH notes.

A completely analogous argument (or consideration of $x\mapsto f(-x)$ instead of $x\mapsto f(x)$) proves that also the set

$$A''_{m,\epsilon}:=\bigl\{x\in(-m,m)\colon f(x)>f(x^-)+\epsilon\bigr\} \tag{28.16}$$

is finite. It now suffices to observe that

$$\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\} = \bigcup_{m\geqslant 1}\bigcup_{n\geqslant 1}\bigl(A'_{m,1/n} \cup A''_{m,1/n}\bigr) \tag{28.17}$$

because $f(x^+) \neq f(x^-)$ implies (in light of f being non-decreasing) $f(x^+) > f(x^-)$ which in turn implies $f(x^+) > f(x) \vee f(x) > f(x^-)$. The claim follows from the fact that a countable union of finite sets is countable. □

Remark 28.6 We note that a rather different-looking proof of Lemma 28.5 was presented in the lecture. Indeed, we dealt directly with the set

$$A_{m,\epsilon} := \{x \in (-m, m) : f(x^+) \neq f(x_-)\} \tag{28.18}$$

by way of the following argument: Assume that $A_{m,\epsilon}$ contains distinct points $x_1, \dots, x_n$ which we label these in an increasing fashion as

$$x_0 := -m < x_1 < \cdots < x_n < m \tag{28.19}$$

we then note the computation

$$\begin{aligned} n\epsilon &\leqslant \sum_{i=1}^n \bigl[f(x_i^+) - f(x_i^-)\bigr] \\ &\leqslant \sum_{i=1}^n \bigl[f(x_i^+) - f(x_{i-1}^+)\bigr] = f(x_n^+) - f(-m^+) \leqslant f(m^+) - f(-m^+) \end{aligned} \tag{28.20}$$

This rules out that $n > [f(m^+) - f(-m^+)]/\epsilon$. Unfortunately, to convert this to a proof of cardinality, we need a mechanism to "pick" such n-tuples of points, which requires a version of Axiom of Choice. The argument we presented in the above proof avoids that.

28.2 Functions of bounded variation.

A natural question that springs to mind is whether the above generalizes beyond monotone functions. This requires introduction of a concept that will be useful in several parts of the course later.

Definition 28.7 (Total variation) *Let $a < b$ be reals. A* partition Π *of interval $[a, b]$ is a finite sequence $\{t_i\}_{i=0}^n$ (for some $n \in \mathbb{N}$) such that*

$$a = t_0 < t_1 < \cdots < t_n = b. \tag{28.21}$$

Given a function $f : [a, b] \to \mathbb{R}$, for each partition $\Pi = \{t_i\}_{i=0}^n$ set

$$V_\Pi\bigl(f, [a, b]\bigr) := \sum_{i=1}^n \bigl|f(t_i) - f(t_{i-1})\bigr| \tag{28.22}$$

The total variation *of f on $[a, b]$ is then the quantity*

$$V\bigl(f, [a, b]\bigr) := \sup_\Pi V_\Pi\bigl(f, [a, b]\bigr), \tag{28.23}$$

where the supremum is over all partitions of $[a, b]$. The supremum takes values in $[0, +\infty]$.

We note that taking the supremum is natural for the following reason:

Lemma 28.8 *If Π, Π' are partitions of $[a,b]$ such that every point of (the sequence defining) Π is contained in (the sequence defining) Π', then $V_\Pi(f,[a,b]) \leqslant V_{\Pi'}(f,[a,b])$.*

We leave the proof of this fact to the reader. We now put forward:

Definition 28.9 (Functions of bounded variation) *A function* $f\colon [a,b] \to \mathbb{R}$ *is of* bounded variation *on* $[a,b]$ *if* $V(f,[a,b]) < \infty$.

It turns out that we have:

Lemma 28.10 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $\forall m \in \mathbb{N}\colon V(f,[-m,m]) < \infty$. Then f has only discontinuities of first kind and $\{x \in \mathbb{R}\colon f(x^+) \neq f(x^-)\}$ is finite or countable.*

However, stating the lemma this way is a bit of a cheat. Indeed, the conclusion follows immediately from the corresponding lemma for monotone functions and the following result dating back to C. Jordan in 1882:

Theorem 28.11 (Jordan decomposition) *Let $f\colon [a,b] \to \mathbb{R}$ be such that $V(f,[a,b]) < \infty$. Then there exist $h,g\colon [a,b] \to \mathbb{R}$ such that*

$$h,g \text{ are non-decreasing } \wedge\ \forall t \in [a,b]\colon\ f(t) = h(t) - g(t) \tag{28.24}$$

Proof. Define $h,g\colon [a,b] \to \mathbb{R}$ by

$$h(t) := V\bigl(f,[a,t]\bigr) \tag{28.25}$$

and

$$g(t) := V\bigl(f,[a,t]\bigr) - f(t) \tag{28.26}$$

(Proving that these take finite values requires showing that $V(f,[a,t]) < \infty$ which we will do momentarily.) Then $f = h - g$ and so it suffices to show that both h and g are non-decreasing.

Let $t,t' \in [a,b]$ be such that $t < t'$. Given a partition $\Pi = \{t_i\}_{i=0}^n$ of $[a,t]$, which entails $t_n = t$, let Π' be the partition $\{t_i'\}_{i=1}^{n+1}$ such that

$$t_{n+1} := t' \ \wedge\ \forall i = 0,\dots,n\colon\ t_i' := t_i. \tag{28.27}$$

Then

$$V\bigl(f,[a,t']\bigr) \geqslant V_{\Pi'}\bigl(f,[a,t']\bigr) = V_\Pi\bigl(f,[a,t]\bigr) + \bigl|f(t') - f(t)\bigr| \tag{28.28}$$

This means that $V(f,[a,t']) - |f(t') - f(t)|$ is an upper bound on $V_\Pi(f,[a,t])$ for every partition Π of $[a,t]$ and so, by the definition of supremum,

$$V\bigl(f,[a,t']\bigr) \geqslant \bigl|f(t') - f(t)\bigr| + V\bigl(f,[a,t]\bigr) \tag{28.29}$$

(Setting $t' := b$ shows that, indeed, $V(f,[a,t]) < \infty$.) Invoking $|f(t') - f(t)| \geqslant 0$ gives

$$h(t') = V\bigl(f,[a,t']\bigr) \geqslant V\bigl(f,[a,t]\bigr) = h(t) \tag{28.30}$$

proving that h is non-decreasing. Similarly we get

$$\begin{aligned} g(t') - g(t) &= V\bigl(f,[a,t']\bigr) - V\bigl(f,[a,t]\bigr) - \bigl[f(t') - f(t)\bigr] \\ &\geqslant \bigl|f(t') - f(t)\bigr| - \bigl[f(t') - f(t)\bigr] \geqslant 0 \end{aligned} \tag{28.31}$$

proving that also g is non-decreasing. □

We call any pair (g,h) of non-decreasing functions such that $f = h - g$ a *Jordan decomposition* of f. As it turns out, being of bounded variation is not only sufficient for existence of a Jordan decomposition but also necessary. Indeed, we have:

Lemma 28.12 *Let $g,h\colon [a,b] \to \mathbb{R}$. Then*

$$V\big(g+h,[a,b]\big) \leqslant V\big(g,[a,b]\big) + V\big(h,[a,b]\big) \tag{28.32}$$

Moreover,

$$h \text{ monotone } \Rightarrow V\big(h,[a,b]\big) = \big|h(b)-h(a)\big|. \tag{28.33}$$

Hence, if g and h are non-decreasing, then $V\big(g-h,[a,b]\big) < \infty$.

We leave the proof of this lemma to the reader. Our next question is whether a Jordan decomposition is unique. A simple answer is that it is not. Indeed, adding a constant or even a non-decreasing function to both g and h does not change the difference of the two while keeping the upward monotonicity of both functions. The right way to ask about the uniqueness is thus whether there is a Jordan decomposition that cannot be further reduced by subtracting a non-decreasing function that vanishes at a from both functions. This is indeed possible but not without additional work. We start with:

Definition 28.13 (Positive and negative variation) *Recall that, for each $z \in \mathbb{R}$ we denote $z_+ := \max\{z,0\}$ and $z_- := \max\{-z,0\}$. Let $f\colon [a,b] \to \mathbb{R}$. The* positive variation *of f on $[a,b]$ is defined by*

$$P\big(f,[a,b]\big) := \sup_{\Pi} \sum_{i=1}^{n} \big(f(t_i) - f(t_{i-1})\big)_+ \tag{28.34}$$

while the negative variation *of f on $[a,b]$ is defined by*

$$N\big(f,[a,b]\big) := \sup_{\Pi} \sum_{i=1}^{n} \big(f(t_i) - f(t_{i-1})\big)_- \tag{28.35}$$

Here, in both formulas, $\{t_i\}_{i=1}^n$ is the sequence corresponding to the partition Π that we take supremum over.

We then have:

Lemma 28.14 *Suppose $f\colon [a,b] \to \mathbb{R}$ be such that $V(f,[a,b]) < \infty$. Then also*

$$P\big(f,[a,b]\big) < \infty \,\wedge\, N\big(f,[a,b]\big) < \infty \tag{28.36}$$

Moreover, we in fact have

$$P\big(f,[a,b]\big) + N\big(f,[a,b]\big) = V\big(f,[a,b]\big) \tag{28.37}$$

and

$$P\big(f,[a,b]\big) - N\big(f,[a,b]\big) = f(b) - f(a) \tag{28.38}$$

Moreover, $t \mapsto P(f,[a,t])$ and $t \mapsto N(f,[a,t])$ are non-decreasing on $[a,b]$.

We leave this to the homework exercise. Using this we now get:

Theorem 28.15 (Minimal Jordan decomposition) *Let $f\colon [a,b]\to\mathbb{R}$ be a function such that $V(f,[a,b])<\infty$. Define $h_0,g_0\colon [a,b]\to\mathbb{R}$ by*

$$h_0(t):=f(a)+P\bigl(f,[a,b]\bigr)\ \wedge\ g_0(t):=N\bigl(f,[a,b]\bigr) \tag{28.39}$$

Then (h_0,g_0) is a Jordan decomposition of f. Moreover, if (h,g) is another Jordan decomposition of f, then $h-h_0$ and $g-g_0$ are both non-decreasing.

Also the proof of this result (which uses similar manipulations as in the proof of Theorem 28.11) is left to a homework exercise.

We close this section with a concept of arc-length, or simply length, of a curve. The reason for bringing this up is that this concept is very similar and, in fact, superior to that of total variation.

Definition 28.16 (Curve and length thereof, rectifiability) *Let (X,ρ) be a metric space. A (parametric)* curve *$\mathcal{C}$ is* $\operatorname{Ran}(f)$ *for $f\colon I\to X$ continuous with $I\subseteq\mathbb{R}$ being an bounded interval. The* (arc-)length *of $\mathcal{C}$ is then defined by*

$$\ell(\mathcal{C}):=\sup_{\Pi}\sum_{i=1}^{n}\rho\bigl(f(t_i),f(t_{i-1})\bigr), \tag{28.40}$$

where the supremum is over all increasing sequences $\Pi:=\{t_i\}_{i=1}^{n}$ from I. The (parametric) curve $\mathcal{C}$ is said to be rectifiable *if $\ell(\mathcal{C})<\infty$.*

Since the arclength is defined using f that parametrizes $\mathcal{C}$, the length of $\mathcal{C}$ depends generally on the parametrization. However, this is not the case if we restrict ourselves to injective f. Unfortunately, this also rules out that $\mathcal{C}$ intersects itself for which a somewhat relaxed notion instead of injectivity is required.

We will return to the arc-lenth after we have discussed differential and integral calculus, which offers a more analytic representation of $\ell(\mathcal{C})$ that one typically encounters in the texts on differential geometry.