26. Intermediate value theorem

We will now move to an application of continuity that has played an important role in the development of the concept itself. We start with the classical result dating back to B. Bolzano in 1817 (although attributions to K. Weierstrass are made as well).

Theorem 26.1 (Intermediate Value Theorem) *Let $a < b$ be reals and $f\colon [a,b] \to \mathbb{R}$ a continuous function with* $\operatorname{Dom}(f) = \mathbb{R}$. *Assume $f(a) \leqslant f(b)$. Then*

$$\forall y \in \bigl[f(a), f(b)\bigr]\, \exists x \in [a,b]\colon\;\; y = f(x) \tag{26.1}$$

To put this in words, a continuous function on a bounded closed interval achieves all values between the values at the endpoints.

Proof. Let $y \in [f(a), f(b)]$. We can assume that $f(a) < f(b)$ (for otherwise $x := a$ will do) and that $y < f(b)$ (for otherwise $x = b$ will do). Define

$$x := \sup\bigl\{z \in [a,b]\colon f(z) \leqslant y\bigr\} \tag{26.2}$$

Since the set contains a (because $f(a) \leqslant y$), we have $x \in [a,b]$. First we claim that $f(x) \leqslant y$. Indeed, the properties of the suprema ensures existence of a sequence $\{z_n\}_{n\in\mathbb{N}}$ with $\forall n \in \mathbb{N}\colon f(z_n) \leqslant y$ such that $z_n \to x$. But the continuity of f then guarantees $f(z_n) \to f(x)$ and so $f(x) \leqslant y$.

Next we claim that $f(x) = y$. Indeed, if NOT then $f(x) < y$. But this means that $x < b$ (for $y \leqslant f(b)$) and by continuity of f at x, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$f\bigl([x, x+\delta)\bigr) \subseteq \bigl(f(x) - \epsilon, f(x) + \epsilon\bigr) \tag{26.3}$$

For $\epsilon := y - f(x)$ this gives $\forall z \in [x, x+\delta)\colon f(z) \leqslant y$ which by $x < b$ contradicts the definition of x. Hence $f(x) = y$ and we are done. $\square$

Remark 26.2 We note that another way to prove this would be to literally limit the proof of the Bolzano-Weierstrass theorem. Indeed, since $y \in [f(a), f(b)]$ we first check whether $y \in [f(a), f(c)]$ or $y \in [f(c), f(b)]$ for $c := \frac{a+b}{2}$. Taking the first interval in which this is true and labeling it $[a_1, b_1]$ we can proceed recursively. This defines two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that $a_0 = a$ and $b_0 = b$ and

$$\forall n \in \mathbb{N}\colon\;\; a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n \,\wedge\, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) \tag{26.4}$$

and, moreover,

$$\forall n \in \mathbb{N}\colon\;\; f(a_n) \leqslant y \leqslant f(b_n). \tag{26.5}$$

As both $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ converge to the same $x \in [a,b]$, continuity of f tells us (by a "Squeeze Theorem" argument) that $y = f(x)$.

A standard application of the Intermediate Value Theorem is:

Corollary 26.3 *Each odd-degree polynomial has a root in $\mathbb{R}$.*

Proof. An odd degree polynomial is a non-zero multiple of $P(x) = x^{2n+1} + Q(x)$ where n is a natural and Q is a polynomial with $\deg(Q) \leqslant 2n$. A simple limit argument for $x \mapsto Q(x)/x^{2n+1}$ as $x \to \pm\infty$ ensures that $P(x) > 0$ for x sufficiently large positive and $P(x) < 0$ for x sufficiently large negative. As P is continuous, the IVT ensures existence of an $x \in \mathbb{R}$ with $P(x) = 0$. $\square$

Another application comes in:

Corollary 26.4 *Let $I \subseteq \mathbb{R}$ be a non-empty bounded closed interval and let $f\colon I \to I$ be a continuous function with* $\mathrm{Dom}(f) = I$. *Then $\exists x \in I\colon\ f(x) = x$.*

We leave the proof of this lemma to homework. Note that the assumption that I is closed is crucial; indeed, $f(x) = x/2$ maps $(0,1)$ into itself without having a fixed point. As it turns out, this corollary is a special case of an important theorem:

Theorem 26.5 (Brower's fixed point theorem) *Let $B := \{x \in \mathbb{R}^d\colon \|x\|_2 \leqslant 1\}$ be the unit (Euclidean) ball in $\mathbb{R}^d$ and let $f\colon B \to B$ a continuous function with* $\mathrm{Dom}(f) = B$. *Then f has a fixed point in B; i.e., $\exists x \in B\colon\ f(x) = x$.*

The proof of this theorem is more complicated as it uses facts from differential geometry. The statement actually extends to continuous functions on compact subsets of $\mathbb{R}^d$ and, in fact, even those in complete linear spaces of infinite dimension. (That version goes by the name Schauder's fixed point theorem.) These fixed point theorems find (at times surprising) applications in mathematics; e.g., optimization, game theory, etc. Their strength is in their relatively general and abstract formulation. The price one pays for this (e.g., compared to Banach's fixed point theorem) is lack of uniqueness and/or method to construct a solution.

Another simple corollary of the Intermediate Value Theorem is:

Corollary 26.6 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and* 2*-periodic; i.e., $\forall x \in \mathbb{R}\colon\ f(x+2) = f(x)$. Then $\exists x \in \mathbb{R}\colon\ f(x+1) = f(x)$.*

Proof. Let $g(x) = f(x+1) - f(x)$. Then g is continuous with

$$g(x+1) = f(x+2) - f(x+1) = f(x) - f(x+1) = -g(x) \tag{26.6}$$

Note that this means that

$$g(0) = 0 = g(1)\ \vee\ g(0) < 0 < g(1)\ \vee\ g(0) > 0 > g(1). \tag{26.7}$$

In all three cases, the Intermediate Value Theorem implies existence of $x \in [0,1]$ such that $g(x) = 0$. The latter is equivalent to $f(x+1) = f(x)$. $\square$

Also this result is a special case of a more general theorem:

Theorem 26.7 (Borsuk-Ulam) *Let $S := \{x \in \mathbb{R}^d\colon \|x\|_1 = 1\}$ be the unit sphere in $\mathbb{R}^d$. (We think of S as a metric space with Eulidean metric inherited from $\mathbb{R}^d$.) Let $f\colon S \to \mathbb{R}$ be a continuous function. Then*

$$\exists x \in S\colon\ f(-x) = f(x) \tag{26.8}$$

Also this theorem requires some elementary facts from differential geometry so we leave its proof to classes dealing with that topic. A popular way to state the result is that at each time, there always exist two points on the opposite sides of the Earth where the temperature is the same. (This includes the statement in Corollary 26.6, provided we interpret a 2-periodic function on $\mathbb{R}$ as a function on a circle of circumference 2.)

While the above results use various facts about metric spaces, the property underlying the Intermediate Value Theorem has not been introduced yet. We fix that in:

Definition 26.8 (Connectedness) *A topological space X is said to be* connected *if*

$$\forall E \subseteq X\colon \quad E \neq \emptyset \,\wedge\, E \text{ open} \,\wedge\, X \smallsetminus E \text{ open} \;\Rightarrow\; E = X \tag{26.9}$$

i.e., if X is not a disjoint union of two non-empty sets that are both open and closed. A set $A \subseteq X$ is said to be connected if it is connected in the relative topology on A.

The definition immediately gives that $\mathbb{R} \smallsetminus \{0\}$ or $\mathbb{R}^2 \smallsetminus \mathbb{R}$ are NOT connected. However, as is intuitive, $\mathbb{R}$ itself or any subinterval thereof are connected although the proof of this requires some work. Indeed, we have:

Lemma 26.9 *Let $a < b$ be reals. Then $[a, b]$ is connected.*

Proof. Let $E \subseteq [a, b]$ be non-empty and relatively open such that also $E^{\mathrm{c}} := [a, b] \smallsetminus E$ is non-empty and relatively open. Without loss of generality (otherwise swap E and E^{c}), assume $a \notin E$. Set $x := \inf(E)$. Then $x \neq a$ because $a \in E^{\mathrm{c}}$ implies that $[a, a + \delta) \subseteq E^{\mathrm{c}}$ for some $\delta > 0$ by the fact that E^{c} is relatively open. But then $x \in E$ implies $(x - \delta', x] \subseteq E$ for some $\delta' > 0$ by the fact that E is relatively open, contradicting the definition of x. Hence we must have $x \in E^{\mathrm{c}}$. But then $(x - \delta'', x + \delta'') \subseteq E^{\mathrm{c}}$ for some δ'' by E^{c} being relatively open which contradicts the fact that $\inf(E)$ is an adherent point of E. Our assumptions thus lead to a contradiction and so $[a, b]$ is connected as desired. □

We now state:

Theorem 26.10 (Intermediate Value Theorem, topological version) *Let X, Y be topological spaces and $f\colon X \to Y$ a continuous function with* $\mathrm{Dom}(f) = X$. *Then*

$$X \text{ connected} \;\Rightarrow\; f(X) \text{ connected} \tag{26.10}$$

Proof. Without loss of generality, we may assume that $f(X) = Y$. Let $E \subseteq Y$ be non-empty and open with $Y \smallsetminus E$ open. Then also $f^{-1}(E)$ is non-empty and, by continuity, open with $f^{-1}(Y \smallsetminus E)$ open. As $f^{-1}(Y \smallsetminus E) = X \smallsetminus f^{-1}(E)$, the connectivity of E implies $f^{-1}(E) = X$. But then $E = f(X) = Y$ thus proving connectivity of $f(X)$. □

To see that this implies Theorem 26.1 note that, by Lemma 26.9 and Theorem 26.10, $f([a, b])$ is connected for any continuous $f\colon [a, b] \to \mathbb{R}$ with $\mathrm{Dom}(f) = [a, b]$. Since $f(a), f(b) \in f([a, b])$, if $y \in [f(a), f(b)]$ obeys $y \notin f([a, b])$, then

$$f\bigl([a, b]\bigr) = \bigl\{z \in f([a, b])\colon z > y\bigr\} \cup \bigl\{z \in f([a, b])\colon z < y\bigr\} \tag{26.11}$$

is a decomposition of $f([a, b])$ into two non-empty relatively open sets, in contradiction with connectivity of $f([a, b])$. It follows that we must have $y \in f([a, b])$ after all meaning that $y = f(x)$ for some $x \in f([a, b])$.

As a final item to discuss, we introduce another notion of connectivity:

Definition 26.11 (Path connectedness) *Let X be a topological space. We say that X is* path connected *if for all $x, y \in X$ there exists a function $f\colon [0, 1] \to X$ such that*

$$\mathrm{Dom}(f) = [0, 1] \,\wedge\, f \text{ continuous} \,\wedge\, f(0) = x \,\wedge\, f(1) = y \tag{26.12}$$

The reason for the name is that the range of f is a continuous curve, or a path, in X. As it turns out, we then have:

Theorem 26.12 *Let X be a topological space. Then*

$$X \text{ path connected } \Rightarrow X \text{ connected} \tag{26.13}$$

The proof of this useful fact is left to homework. Note that the result immediately implies that $\mathbb{R}$ or any interval in $\mathbb{R}$ is connected. (This is a bit of a cheat because the proof of Theorem 26.12 actually follows very much that of Lemma 26.9.) To give a non-trivial example where Theorem 26.12 is of independent value, note that $\mathbb{R}^3 \smallsetminus \mathbb{R}$ is path connected and thus connected. Proving connectedness of $\mathbb{R}^3 \smallsetminus \mathbb{R}$ directly seems to be difficult and path connectedness makes it rather easy.

We remark that the converse to (26.13) fails in general. To give a counterexample, let $\{a_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive reals with $a_0 := 1$ and $a_n \to 0$. Define the function $h\colon (0,1] \to [0,1]$ by

$$h(x) := \begin{cases} 0, & \text{if } \exists n \in \mathbb{N}\colon x = a_{2n}, \\ 1, & \text{if } \exists n \in \mathbb{N}\colon x = a_{2n+1}, \\ \text{linear}, & \text{otherwise.} \end{cases} \tag{26.14}$$

Now take

$$A := (\{0\} \times [0,1]) \cup \{(x, h(x))\colon x \in (0,1]\} \tag{26.15}$$

regarded as a subset of $\mathbb{R}^2$ with the Euclidean metric. Then A is connected (more or less because every two disjoint closed subsets of $\mathbb{R}^2$ have a positive distance between them) but not path connected because A contains no continuous path between $(x, h(x))$ (for $x \in (0,1]$) and a point of the form $(0, y)$ — indeed, any such path would contain sequences of points converging to $(0,0)$ and $(0,1)$, which is impossible under continuity (of the function defining the path) which dictates convergence to $(0, y)$.