25. Uniform and Cauchy continuity

We have seen above that continuous functions are exactly those that convert convergent sequences to convergent sequences. A natural variation on this is: What functions convert Cauchy sequences to Cauchy sequences. We will give these a name:

Definition 25.1 (Cauchy continuity) *Let $f\colon X \to Y$ be a function between metric spaces X and Y. We say that f is* Cauchy continuous *if*

$$\forall \{x_n\}_{n\in\mathbb{N}} \in \operatorname{Dom}(f)^{\mathbb{N}}\colon\ \{x_n\}_{n\in\mathbb{N}} \text{ Cauchy (in } X) \ \Rightarrow\ \{f(x_n)\}_{n\in\mathbb{N}} \text{ Cauchy (in } Y) \tag{25.1}$$

i.e., if f turns Cauchy sequences to Cauchy sequences.

We immediately note:

Lemma 25.2 (AC) *A Cauchy continuous function is continuous.*

Proof. Let $f\colon X \to Y$ be Cauchy continuous and let $x \in \operatorname{Dom}(f)$. Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \in \operatorname{Dom}(f)^{\mathbb{N}}$ such that $x_n \to x$ and define $\{y_n\}_{n\in\mathbb{N}}$ by $y_{2n} := x_n$ and $y_{2n+1} := x$. Then also $y_n \to x$. But then $\{f(y_n)\}_{n\in\mathbb{N}}$ is Cauchy which means

$$\rho_Y\big(f(x_n), f(x)\big) = \rho_Y\big(f(y_{2n}), f(y_{2n+1})\big) \to 0 \tag{25.2}$$

Hence $f(x_n) \to f(x)$. By Theorem 24.7 (which is where we need to call upon the Axiom of Choice), f is continuous at x. □

However, the converse does not hold. Indeed consider a function $f\colon \mathbb{R} \smallsetminus \{0\} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1, & \text{if } x > 0,\\ 0, & \text{if } x < 0, \end{cases} \tag{25.3}$$

Then f is continuous yet not Cauchy continuous because the sequence $x_n := (-2)^{-n}$ is Cauchy yet $\{f(x_n)\}_{n\in\mathbb{N}}$ is NOT.

The concept of Cauchy continuity is somewhat special and in practice we typically use different concepts that address directly continuity in somewhat quantitative way. The simplest of these is:

Definition 25.3 (Uniform continuity) *A function $f\colon X \to Y$ between metric spaces (X, ρ_X) and (Y, ρ_Y) is said to be* uniformly continuous *if*

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x, y \in X\colon\ \rho_X(x,y) < \delta \ \Rightarrow\ \rho_Y\big(f(x), f(y)\big) < \epsilon \tag{25.4}$$

As it turns out, this differs from the definition of continuity in a rather inconspicuous yet very important way. Indeed, a function $f\colon X \to Y$ is continuous if

$$\forall \epsilon > 0\, \forall x \in X\, \exists \delta > 0\, \forall y \in X\colon\ \rho_X(x,y) < \delta \ \Rightarrow\ \rho_Y\big(f(x), f(y)\big) < \epsilon \tag{25.5}$$

which differs from (25.4) by a swap of $\exists\delta > 0$ and $\forall x \in X$ quantifiers. This swap amounts to the fact that δ in (25.5) may depend on x yet in (25.4) one δ must work for all x (and y) simultaneously. This immediately gives:

Lemma 25.4 *A uniformly continuous function is continuous.*

Again, the converse to this fails as witnessed by the example

$$f(x) := x^2 \tag{25.6}$$

with $\mathrm{Dom}(f) = \mathbb{R}$. Indeed, $(x+\delta)^2 - x^2 = 2x\delta + \delta^2$ which cannot be made smaller than ϵ without restricting the size of x. We in fact have a stronger conclusion:

Lemma 25.5 *A uniformly continuous function is Cauchy continuous.*

Proof. Let $f\colon X \to Y$ be uniformly continuous. Given $\epsilon > 0$ let $\delta > 0$ be such that $\rho_X(x,y) < \delta$ implies $\rho_Y(f(x), f(y)) < \epsilon$. Given a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$, there exists $n_0 \geqslant 0$ such that $\rho_X(x_n, x_m) < \delta$ for all $n, m \geqslant n_0$. But then $\rho_X(f(x_n), f(x_m)) < \epsilon$ for all $n, m \geqslant n_0$. As this holds for all $\epsilon > 0$, $\{f(x_n)\}_{n\in\mathbb{N}}$ is Cauchy as desired. $\square$

The example (25.6) shows that the converse to this does not hold; indeed, the function there is Cauchy continuous yet not uniformly continuous. However, this would not be the case if we restrict that f to any bounded subset of $\mathbb{R}$. Indeed, we have:

Lemma 25.6 *Let X and Y be metric space with X compact. The for all $f\colon X \to Y$:*

$$f \text{ continuous } \Rightarrow f \text{ uniformly continuous} \tag{25.7}$$

(Note that, thanks to Lemma 25.4, we can even write $\Leftrightarrow$ in (25.7)*.)*

Proof. Assume X to be compact and let $f\colon X \to Y$ be continuous. Given $\epsilon > 0$, for each $x \in X$ let

$$\Delta_x := \bigl\{\delta \in (0,\infty)\colon\ \mathrm{diam}\, f(B_X(x, 2\delta)) < \epsilon\bigr\} \tag{25.8}$$

The continuity of f at x ensures that $\Delta_x \neq \emptyset$ for all $x \in X$. This, along with the fact that $x \in B_X(x,\delta)$ once $\delta > 0$ shows that $\{B_X(x,\delta)\colon x \in X \wedge \delta \in \Delta_x\}$ is an open cover X. The assumed compactness of X ensures existence of $n \in \mathbb{N}$, $z_0, \dots, z_n \in X$ and $\delta_0, \dots, \delta_n \in (0,\infty)$ such that

$$X = \bigcup_{i=0}^{n} B_X(z_i, \delta_i) \ \wedge\ \forall i = 0, \dots, n\colon \delta_i \in \Delta_{z_i} \tag{25.9}$$

Define $\delta := \min_{i=0,\dots,n} \delta_i$ and note that $\delta > 0$. Then pick any $x, y \in X$ with $\rho_X(x,y) < \delta$ and observe that for

$$i := \min\bigl\{j = 0, \dots, n\colon x \in B(z_j, \delta_j)\bigr\} \tag{25.10}$$

we have

$$\rho(y, z_i) \leqslant \rho(x,y) + \rho(x, z_i) < \delta + \delta_i \leqslant 2\delta_i \tag{25.11}$$

and, since $\rho_X(x, z_i) < \delta \leqslant \delta_i$, we have $x, y \in B_X(x_i, 2\delta_i)$. The fact that $\delta_i \in \Delta_{z_i}$ then gives

$$\rho_Y\bigl(f(x), f(y)\bigr) \leqslant \mathrm{diam}\, f\bigl(B_X(z_i, 2\delta_i)\bigr) < \epsilon. \tag{25.12}$$

As $x, y \in X$ were arbitrary points with $\rho_X(x,y) < \delta$, this proves uniform continuity of f on X. $\square$

The assumption of compactness of X can be relaxed and the conclusion strengthened to the following form:

Lemma 25.7 *Let X and Y be metric space with X totally bounded. The for all $f\colon X \to Y$:*

$$f \text{ Cauchy continuous } \Rightarrow f \text{ uniformly continuous} \tag{25.13}$$

We leave the proof of this lemma to homework. A key reason for dealing with all of these concepts comes in:

Theorem 25.8 (AC)(Existence/uniqueness of continuous extension) *Let X and Y be metric spaces, $A \subseteq X$ a non-empty set and $f\colon A \to Y$ a function with* $\operatorname{Dom}(f) = A$. *Assume:*

(1) *A is dense in X; i.e., $\overline{A} = X$,*
(2) *Y is complete*
(3) *f is uniformly continuous (or even just Cauchy continuous).*

Then there exists a continuous function $\bar f\colon X \to Y$ such that

$$\forall x \in X\colon\ \bar f(x) = f(x) \tag{25.14}$$

Moreover, $\bar f$ is unique in the sense that if $g\colon X \to Y$ is a continuous function with $\operatorname{Dom}(g) = X$ *and $g = f$ on A, then $g = \bar f$ on X.*

Proof. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces as is the setting of the theorem and assume that A and Y are such that the conditions (1-2) above apply. Let $f\colon A \to Y$ be a function with $\operatorname{Dom}(f) = A$ and assume that f is Cauchy continuous (which is implied by uniform continuity). We start with the construction of $\bar f$.

Recall that $f\colon X \to Y$ with $\operatorname{Dom}(f) = A$ is technically a relation $G \subseteq X \times Y$ of the specific form

$$G := \bigl\{(x, f(x)) \in X \times Y\colon x \in A\bigr\} \tag{25.15}$$

that we typically refer to as the graph of f. Next note that $X \times Y$ is a metric space relative to the metric

$$\rho\bigl((x,y),(\tilde x,\tilde y)\bigr) = \rho_X(x,\tilde x) + \rho_Y(y,\tilde y) \tag{25.16}$$

This permits us to consider the closure $\overline{G}$ of G in $X \times Y$. We now claim:

Claim 1: $\overline{G}$ is the graph of a function

Indeed, suppose $(x,y),(x,\tilde y) \in \overline{G}$. Then the fact that these are adherent points of G, which is the graph of f, imply (by the AC) existence of sequences $\{x_n\}_{n\in\mathbb{N}}, \{\tilde x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ such that

$$\bigl(x_n, f(x_n)\bigr) \longrightarrow (x,y) \ \wedge\ \bigl(\tilde x_n, f(\tilde x_n)\bigr) \longrightarrow (x,\tilde y) \tag{25.17}$$

with the convergences in the metric space $(X \times Y, \rho)$. But this means that $x_n \to x$ and $\tilde x_n \to x$ and thus also that the sequence $\{z_n\}_{n\in\mathbb{N}}$ defined by

$$z_{2n} := x_n \ \wedge\ z_{2n+1} = \tilde x_n \tag{25.18}$$

obeys $z_n \to x$. But the fact that f is Cauchy then implies that $\{f(z_n)\}_{n\in\mathbb{N}}$ is Cauchy and, since also $f(x_n) \to y$ and $f(\tilde x_n) \to \tilde y$, shows

$$\rho_Y(y,\tilde y) = \lim_{n\to\infty} \rho_Y\bigl(f(x_n), f(\tilde x_n)\bigr) = \lim_{n\to\infty} \rho_Y\bigl(f(z_{2n}), f(z_{2n+1})\bigr) = 0 \tag{25.19}$$

This proves that

$$\forall (x,y),(x,\tilde y) \in \overline{G}\colon\ y = \tilde y \tag{25.20}$$

and so $\overline{G}$ is the graph of a function. Let us denote this function by $\bar f$. Next we note:

Claim 2: $\operatorname{Dom}(\bar f) = X$

To prove this, let $x \in X$. Then by (1) above x is adherent to A and so (by the AC) there exists $\{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $x_n \to x$. The Cauchy continuity of f implies that $\{f(x_n)\}_{n\in\mathbb{N}}$

is a Cauchy sequence and so, by the completeness of Y assumed in (2) above, there exists $y \in Y$ such that $f(x_n) \to Y$. But then $(x_n, f(x_n)) \to (x, y)$ in $X \times Y$ proving that $(x, y) \in \overline{G}$ and thus $x \in \mathrm{Dom}(\bar{f})$.

Claim 3: $\bar{f}$ is continuous Since $G \subseteq \overline{G}$, the function $\bar{f}$ is an extension of f to all points of X. To prove continuity let $x \in X$ assume for contradiction that there is $\{x_n\}_{n\in\mathbb{N}} \in X^N$ such that

$$x_n \to x \;\wedge\; \forall n \in \mathbb{N}\colon\; \rho_Y(\bar{f}(x_n), \bar{f}(x)) \geqslant \epsilon \tag{25.21}$$

By the construction of $\overline{G}$, there exists $\{\tilde{x}_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}\colon \rho_X(x_n, \tilde{x}_n) < 2^{-n} \;\wedge\; \rho_Y\big(\bar{f}(x_n), f(\tilde{x}_n)\big) < \epsilon/2 \tag{25.22}$$

But then $\tilde{x}_n \to x$ and, by Cauchy continuity of f and completeness of Y, there exists $y \in Y$ such that $f(\tilde{x}_n) \to y$. It follows that $\bar{f}(x) = y$ yet the second halves of (25.21–25.22), $\rho_Y(y, \bar{f}(x)) \geqslant \epsilon/2$, and thus $y \neq \bar{f}(x)$. As $(x, \bar{f}(x)), (x, y) \in \overline{G}$, this contradicts that $\overline{G}$ is the graph of a function. It follows that $\bar{f}$ is continuous.

For the uniqueness it suffices to note that two continuous functions defined on a closure of the set A agree once they agree on A. We leave this detail to the reader. $\square$

We remark that uniform continuity allows us to avoid the use of the Axiom of Choice in the proof of Claim 1 but does not seem to do that for the rest of the claim, due to the fact that the completeness of Y requires working with Cauchy sequences to begin with.

To demonstrate the power of this result, let us prove one more time the existence of exponential function:

Lemma 25.9 *Let $a > 0$ be real and recall that $f(x) := a^x$ is well defined for all $x \in \mathbb{Q}$. Then f is Cauchy continuous and thus extends continuously to a unique continuous function on $\mathbb{R}$ (still written as $x \mapsto a^x$).*

Proof. Assume without (much) loss of generality that $a \geqslant 1$. We will rely on the fact that $a^{x+y} = a^x a^y$ which is checked algebraically for $x, y \in \mathbb{Q}$. This shows

$$a^y - a^x = a^x(a^{y-x} - 1) \tag{25.23}$$

Next note that for each $\delta \in (0,1)$ there is $N(\delta) \in \mathbb{N}$ such that $1 - \delta < a^{\frac{1}{N(\delta)+1}} < 1 + \delta$. Indeed, if note than either $a > (1+\delta)^{1+n}$ for all $n \geqslant 0$ or $a < (1-\delta)^{n+1}$ for all $n \geqslant 0$ which is impossible due to the fact that a is positive and finite.

Now let $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ be a Cauchy sequence. This sequence is bounded by, say, $M \in \mathbb{N}$. Pick $\epsilon > 0$ and set $\delta := \epsilon a^{-M}$. The fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy implies

$$\exists n_0 \in \mathbb{N}\, \forall n, m \geqslant n_0\colon\; |x_n - x_m| < \frac{1}{N(\delta) + 1} \tag{25.24}$$

But then for $m, n \geqslant n_0$ the above shows

$$\big|f(x_m) - f(x_n)\big| = a^{x_m}|a^{x_n - x_m} - 1| < a^M \delta = \epsilon \tag{25.25}$$

thus proving that $\{f(x_n)\}_{n\in\mathbb{N}}$ is Cauchy. Theorem 25.7 now yields the desired unique continuous extension of f to all of $\mathbb{R}$. $\square$

To finish the discussion of uniform continuity, we introduce a couple of standard notions that give a quantitative form of the dependence of δ on ϵ in (25.4).

Definition 25.10 (Lipschitz/Hölder functions) *Let $f\colon X \to Y$ be a function between metric spaces (X,ρ_X) and (Y,ρ_Y). We say that f is* Lipschitz continuous *if*

$$\exists\lambda > 0\forall x,y \in X\colon\;\; \rho_Y\big(f(x),f(y)\big) \leqslant \lambda\rho_X(x,y) \tag{25.26}$$

and, given $\alpha > 0$, is α-Hölder continuous if

$$\exists\lambda > 0\forall x,y \in X\colon\;\; \rho_Y\big(f(x),f(y)\big) \leqslant \lambda\rho_X(x,y)^\alpha \tag{25.27}$$

Note that Lipschitz continuity is a special case of Hölder continuity (corresponding to $\alpha := 1$) but the vernacular is used in this form throughout mathematics. Both concepts give a quantitative form of dependence of δ on ϵ in (25.4); namely, $\epsilon = \lambda\delta$ for (25.26) and $\epsilon = \lambda\delta^\alpha$ in (25.27). A Lipschitz/Hölder continuous function is uniformly continuous and thus also continuous.

The smallest constant λ that one can put into these expression is sometimes called the Lipschitz/Hölder norm. The cases with $\alpha \in (0,1]$ are most natural because $x,y \mapsto \rho_X(x,y)^\alpha$ is a metric. As we will see, this is all there is for functions $\mathbb{R} \to \mathbb{R}$; indeed, an α-Hölder function $\mathbb{R} \to \mathbb{R}$ (relative to Euclidean metric) for $\alpha > 1$ is necessarily constant.

Both properties above try to estimate the distance of function values by a function of the distance of the arguments. This naturally leads to the following generalization:

Definition 25.11 (Modulus of continuity) *Given a function $f\colon X \to Y$ between metric spaces (X,ρ_X) and (Y,ρ_Y) and a continuous non-decreasing map $\omega\colon [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ and $\omega(t) > 0$ for $t > 0$, we say that ω is a modulus of continuity of f if*

$$\forall x,y \in X\colon\;\; \rho_Y\big(f(x),f(y)\big) \leqslant \omega\big(\rho_X(x,y)\big) \tag{25.28}$$

So $\omega(t) = \lambda t$ in (25.26) and $\omega(t) = \lambda t^\alpha$ in (25.27). The objective above is of course to find a "best" function ω that works for the given f; if for each $\lambda \in (0,1)$ there is a pair $x,y \in X$ with $\rho_X(x,y) > 0$ and $\rho_Y(f(x),f(y)) \geqslant \lambda\omega(\rho_X(x,y))$, then we sometimes talk about *the* modulus of continuity of f. (Other interpretations of this term do exist, though.) Another way to study the behavior of functions is using:

Definition 25.12 (Oscillation) *Let $f\colon X \to Y$ be a function between metric spaces (X,ρ_X) and (Y,ρ_Y) and let $A \subseteq X$. The* oscillation of f on A *is the function $r \mapsto \operatorname{osc}_f(A,r)$ where*

$$\operatorname{osc}_f(A,r) := \sup\Big\{\rho_Y\big(f(x),f(y)\big)\colon x,y \in A \,\wedge\, \rho_X(x,y) < r\Big\} \tag{25.29}$$

Here r is restricted to positive reals.

We note that all of these concepts are just ways to express properties of a function in somewhat more condensed way.