24. Continuity

Much of attention in analysis is paid to functions that have some level of regularity. Here we start with the simplest instance of regularity called continuity. Throughout we assume familiarity with metric spaces, topological spaces and convergence of sequences.

24.1 Continuity in metric spaces.

Continuity of a function is often described informally as the ability to "draw the graph of a function without lifting the pen from the paper." A slightly more formal way to put is that "a small change in the argument results in a small change of the function value." Relying on the formalism of metric spaces to quantity the "change," this leads to:

Definition 24.1 (Continuity in metric spaces) *Let (X,ρ_X) and (Y,ρ_Y) be metric spaces and $f\colon X\to Y$ a function (with* $\text{Dom}(f)$ *not necessarily equal to X). Let $x_0\in\text{Dom}(f)$. We say that f is* continuous at x_0 *if*

$$\forall\epsilon>0\,\exists\delta>0\,\forall x\in\text{Dom}(f)\colon\ \rho_X(x,x_0)<\delta\;\Rightarrow\;\rho_Y\bigl(f(x),f(x_0)\bigr)<\epsilon. \tag{24.1}$$

Moreover, we say that f is continuous *if $\forall x\in\text{Dom}(f)\colon f$ is continuous at x.*

A couple of remarks are in order:

(1) Allowing that $\text{Dom}(f)\neq X$ is superfluous because the above definition of continuity can be directly phrased using the relative metric on $\text{Dom}(f)$. We will thus take $\text{Dom}(f)=X$ in the subsequent discussion. However, this is not to say that domain questions are not of interest; e.g., take $f:\mathbb{Q}\to\mathbb{R}$ defined by

$$f(x):=\begin{cases}1, & \text{if } x>\sqrt{2},\\ 0, & \text{if } x<\sqrt{2},\end{cases} \tag{24.2}$$

which is continuous on $\mathbb{Q}$ in spite of a "jump" at $\sqrt{2}$.

(2) Denoting the open balls in X and Y by

$$\begin{aligned}B_X(x_0,r)&:=\{x\in X\colon\rho_X(x,x_0)<r\}\\ B_Y(y_0,r)&:=\{y\in Y\colon\rho_X(y,y_0)<r\},\end{aligned} \tag{24.3}$$

respectively, the implication in (24.1) is *equivalent* to

$$f\bigl(B_X(x_0,\delta)\bigr)\subseteq B_Y\bigl(f(x_0),\epsilon\bigr). \tag{24.4}$$

Continuity of f at x thus means that each open ball centered at $f(x)$ contains the image of a sufficiently small open ball centered at x.

(3) We cast the definition directly in general metric spaces even though most of our attention will be devoted to real-valued functions of $\mathbb{R}$ or $\mathbb{R}^d$-valued variables. This is amounts to setting $Y:=\mathbb{R}$ with ρ_Y being the Euclidean metric, $\rho_Y(y,y'):=|y-y'|$. Since all norm-metric in $\mathbb{R}^d$ are comparable, a function $f\colon\mathbb{R}^d\to\mathbb{R}$ is continuous at x or NOT regardless of the choice of the norm-metric on $\mathbb{R}^d$.

As it turns out, every metric space (X,ρ_X) is the domain of at least three continuous maps. First, the constant map $f\colon X\to Y$ given by $f(x)=y_0$, for any choice of Y and $y_0\in Y$ for which (24.4) holds because $f(B_X(x_0,\delta))=\{f(x_0)\}$ regardless of $\delta>0$. Second, the identity map $f\colon X \to X$ given by $f(x) := x$, for which (24.1) holds with $\delta := \epsilon$. Third, the distance-to-a-point map $f\colon X \to \mathbb{R}$ given for any $x' \in X$ by

$$f(x) := \rho_X(x', x), \tag{24.5}$$

which obeys

$$\rho_{\mathbb{R}}\big(f(x), f(x_0)\big) := \big|f(x) - f(x_0)\big| = \big|\rho_X(x', x) - \rho_X(x', x_0)\big| \leqslant \rho_X(x, x_0) \tag{24.6}$$

thanks to the triangle inequality for ρ_X, thus showing (24.1) with $\delta := \epsilon$. A variation on the latter map is distance-to-a-set map $f(x) := \inf\{\rho_X(x,y)\colon y \in A\}$ for any non-empty $A \subseteq X$. Such maps naturally arise in various contects.

24.2 "Rules" for continuity.

Once we specialize to real-valued functions, the set of continuous functions becomes considerably richer due to the fact that continuity is preserved by the basic arithmetic operations. This leads to various "Rules" for continuity that we discusse next:

Lemma 24.2 (Sum Rule for continuity) *Let (X, ρ_X) be a metric space and $f, g\colon X \to \mathbb{R}$ be functions with* $\operatorname{Dom}(f) = \operatorname{Dom}(g) = X$. *Let $x_0 \in X$ and assume that f and g are continuous at x_0. Then also the their $f + g$ defined by*

$$(f+g)(x) := f(x) + g(x) \tag{24.7}$$

is continuous at x_0.

Proof. Pick $\epsilon > 0$. By continuity of f at x_0, there exists $\delta > 0$ be such that

$$\forall x \in X\colon\ \rho_X(x, x_0) < \delta \Rightarrow \big|f(x) - f(x_0)\big| < \epsilon/2 \tag{24.8}$$

The continuity of g at x_0 in turn yields a $\delta' > 0$ such that

$$\forall x \in X\colon\ \rho_X(x, x_0) < \delta' \Rightarrow \big|g(x) - g(x_0)\big| < \epsilon/2 \tag{24.9}$$

Then for $\delta'' := \min\{\delta, \delta'\}$ and any $x' \in X$, the assumption $\rho_X(x, x_0) < \delta''$ implies

$$\begin{aligned} \big|(f+g)(x) - (f+g)(x_0)\big| &= \Big|f(x) - f(x_0) + \big(g(x) - g(x_0)\big)\Big| \\ &\leqslant \big|f(x) - f(x_0)\big| + \big|g(x) - g(x_0)\big| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned} \tag{24.10}$$

thus proving continuity of $f + g$ at x_0. □

Lemma 24.3 (Product Rule for continuity) *Let (X, ρ_X) be a metric space and $f, g\colon X \to \mathbb{R}$ functions with* $\operatorname{Dom}(f) = \operatorname{Dom}(g) = X$. *Let $x_0 \in X$ and assume that f and g are continuous at x_0. Then also the product function $f \cdot g$ defined by*

$$(f \cdot g)(x) := f(x) \cdot g(x) \tag{24.11}$$

is continuous at x_0.

Proof. The proof is similar to the Sum Rule; we only have to be slightly more clever about our choice of δ. Indeed, assume that $\epsilon > 0$ is given and use continuity of f at x to find $\delta > 0$ such that

$$\forall x \in X\colon\ \rho_X(x, x_0) < \delta \Rightarrow \big|f(x) - f(x_0)\big| < \min\Big\{\epsilon, \frac{1}{2}\frac{\epsilon}{\epsilon + |g(x_0)|}\Big\}. \tag{24.12}$$

Similarly, use the continuity of g at x_0 to find $\delta > 0$ such that

$$\forall x \in X\colon\ \rho_X(x,x_0) < \delta' \ \Rightarrow\ \bigl|g(x)-g(x_0)\bigr| < \frac{1}{2}\,\frac{\epsilon}{\epsilon+|f(x_0)|}. \tag{24.13}$$

Set again $\delta'' := \min\{\delta,\delta'\}$. Then for all $x \in X$ with $\rho_X(x,x_0) < \delta''$ we have

$$\begin{aligned}\bigl|(f\cdot g)(x)-(f\cdot g)(x_0)\bigr| &= \Bigl|f(x)\bigl(g(x)-g(x_0)\bigr)+g(x_0)\bigl(f(x)-f(x_0)\bigr)\Bigr| \\ &\leqslant \bigl|f(x)\bigr|\bigl|g(x)-g(x_0)\bigr| + \bigl|g(x_0)\bigr|\bigl|f(x)-f(x_0)\bigr| \\ &< \frac{1}{2}\,\frac{\epsilon|f(x)|}{\epsilon+|f(x_0)|} + \frac{1}{2}\,\frac{\epsilon|g(x_0)|}{\epsilon+|g(x_0)|} \leqslant \epsilon/2+\epsilon/2 = \epsilon,\end{aligned} \tag{24.14}$$

where in the last inequality we noted that, by (24.12),

$$\bigl|f(x)\bigr| \leqslant \bigl|f(x_0)\bigr| + \bigl|f(x)-f(x_0)\bigr| \leqslant \bigl|f(x_0)\bigr| + \epsilon \tag{24.15}$$

which then leads to the resulting bound. □

Lemma 24.4 (Quotient Rule for continuity) *Let (X,ρ_X) be a metric space and $f,g\colon X\to\mathbb{R}$ functions with* Dom(f) = Dom(g) = X. *Let $x_0 \in X$ and assume that f and g are continuous at x_0 and $g(x_0) \neq 0$. Then also their quotien f/g with* Dom$(f/g) := \{x \in X\colon g(x) \neq 0\}$ *defined by*

$$(f/g)(x) := \frac{f(x)}{g(x)} \tag{24.16}$$

is continuous at x_0.

Proof. We first prove this for $f := 1$. Assume that g is continuous at x_0 with $g(x_0) \neq 0$. Pick $\epsilon > 0$ and let $\delta > 0$ be such that

$$\forall x \in X\colon\ \rho_X(x,x_0) < \delta \ \Rightarrow\ \bigl|g(x)-g(x_0)\bigr| < \min\Bigl\{\frac{1}{2}\epsilon g(x_0)^2, \frac{1}{2}|g(x_0)|\Bigr\}. \tag{24.17}$$

Then for any x with $\rho_X(x,x_0) < \delta$,

$$\bigl|g(x)\bigr| \geqslant \bigl|g(x_0)\bigr| - \bigl|g(x)-g(x_0)\bigr| \geqslant \bigl|g(x_0)\bigr| - \bigl|g(x_0)\bigr|/2 = |g(x_0)|/2 \tag{24.18}$$

which by $g(x_0) \neq 0$ implies $x \in \mathrm{Dom}(1/g)$. Moreover,

$$\Bigl|\frac{1}{g(x)} - \frac{1}{g(x_0)}\Bigr| = \frac{|g(x)-g(x_0)|}{|g(x)||g(x_0)|} \leqslant 2\frac{|g(x)-g(x_0)|}{g(x_0)^2} < \epsilon \tag{24.19}$$

proving the continuity of $1/g$ at x_0. This extends to function f/g by the Product Rule proved in Lemma 24.3. □

As a consequence we get:

Corollary 24.5 (Continuity of polynomials and rational functions) *All polynomials are continuous on all of $\mathbb{R}$. Any rational function of the form $R(x) := P(x)/Q(x)$ where P and Q are polynomials is continuous on its domain* Dom$(R) := \{x \in \mathbb{R}\colon Q(x) \neq 0\}$.

Proof. The constant maps $f(x) = c$ and the identity map $f(x) := x$ are continuous by our earlier observations. Any power $g(x) := x^n$ is then continuous by applying the product rule inductively. Constant multiples of powers are continuous by the product rule as well; the sum rule then implies continuity of all polynomials. The quotient rule extends this to the continuity of rational functions on their domain. □

We remark that, for the Sum Rule, we did not need that f and g are real valued; it was enough to assume that they take values in a linear vector space with a norm-metric. For the Product Rule we needed that f and g take values in a *normed algebra* such that the norm of the product of two elements is bounded by the product of the norms; i.e., $\|a \cdot b\| \leqslant \|a\| \, \|b\|$.

Another operation that is useful for generating continuous functions (or checking continuity of functions in general) is composition:

Lemma 24.6 (Composition Rule) *Let (X, ρ_X), (Y, ρ_Y) and (Z, ρ_Z) be metric spaces and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Let $x_0 \in \text{Dom}(f)$ be such that $f(x_0) \in \text{Dom}(g)$. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f(x) := g(f(x))$ is continuous at x_0.*

Proof. Pick $\epsilon > 0$ and let $\eta > 0$ be such that $\rho_Y(y, f(x_0)) < \eta$ (and $y \in \text{Dom}(g)$) implies $\rho_Z(g(y), g(f(x_0))) < \epsilon$. Given this η, let $\delta > 0$ be such that $\rho_X(x, x_0) < \delta$ (and $x \in \text{Dom}(f)$) implies $\rho_Y(f(x), f(x_0)) < \eta$. But then for all such x we also get $\rho_Z(g(f(x)), g(f(x_0))) < \epsilon$ thus implying continuity of $g \circ f$ at x_0. □

24.3 Alternative characterizations.

Two alternative characterizations of continuity appear in the literature (sometimes even as alternative definitions) and are often called upon in practice. The first one of these is based on the concept of limit of sequences that we discussed at length earlier.

Theorem 24.7 (AC)(Sequential characterization) *Let (X, ρ_X) and (Y, ρ_Y) be metric spaces and let $f\colon X \to Y$ be a function with $\text{Dom}(f) = X$. Then for all $x \in X$:*

$$f \text{ is continuous at } x \;\Leftrightarrow\; \Big(\forall \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}\colon\; x_n \to x \;\Rightarrow\; f(x_n) \to f(x)\Big) \tag{24.20}$$

where the convergences are relative to ρ_X on the left and ρ_Y on the right.

Proof. We start with $\Rightarrow$. Assume that f is continuous at x and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence. Given $\epsilon > 0$, the continuity yields a $\delta > 0$ such that $\rho_X(x_n, x) < \delta$ implies $\rho_Y(f(x), f(x_n)) < \epsilon$. Under the assumption $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that $n \geqslant n_0$ implies $\rho_X(x_n, x) < \delta$. Summarizing, this shows that given $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $n \geqslant n_0$ implies $\rho_Y(f(x_n), f(x)) < \epsilon$, thus proving $f(x_n) \to f(x)$ in (Y, ρ_Y).

The proof of $\Leftarrow$ is done by contrapositive. Suppose f is NOT continuous at x. Then, using that (24.1) and (24.4) are equivalent,

$$\exists \epsilon > 0 \,\forall \delta > 0\colon f\big(B_X(x, \delta)\big) \smallsetminus B_Y\big(f(x), \epsilon\big) \neq \emptyset \tag{24.21}$$

Specializing to $\delta \in \{2^{-n}\colon n \in \mathbb{N}\}$, the Axiom of Choice allows us to pick a sequence $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}\colon x_n \in B_X(x, 2^{-n}) \wedge f(x_n) \notin B_Y\big(f(x), \epsilon\big). \tag{24.22}$$

But this means that $x_n \to x$ while $f(x_n) \nrightarrow f(x)$, proving the negation of the statement on the right of (24.20) — as is required in the proof by contrapositive. □

A simpler but memorable way to state the conclusion of Theorem 24.7 is:

Corollary 24.8 (AC) *A function (between two metric spaces) is continuous if and only if it turns convergent sequences into convergent sequences.*

The reliance on the Axiom of Choice is annoying but this is not of much loss is practice. Indeed, the sequential characterization is typically used to *disprove* continuity by demonstrating a sequence $x_n \to x$ with $f(x_n) \nrightarrow f(x)$. To give an example, consider the function in (24.2) albeit now defined on all of $\mathbb{R}$:

$$f(x) := \begin{cases} 1, & \text{if } x \geqslant \sqrt{2}, \\ 0, & \text{if } x < \sqrt{2}, \end{cases} \tag{24.23}$$

Then taking $x_n := \sqrt{2} - 2^{-n}$ gives $f(x_n) = 0$ so $f(x_n) \to 0$, yet $x_n \to \sqrt{2}$ and $f(\sqrt{2}) = 1$. Hence f is NOT continuous at $x = \sqrt{2}$ by $\Rightarrow$ in (24.20) (for if it were continuous than we would have $f(x_n) \to f(\sqrt{2})$).

Another useful characterization of continuity comes in:

Theorem 24.9 (Topological characterization of continuity) *Let (X, ρ_X) and (Y, ρ_Y) be metric spaces and let $f\colon X \to Y$ be a function with* $\mathrm{Dom}(f) = X$. *Then the following are equivalent:*

(1) f is continuous,
(2) $\forall O \subseteq Y\colon\ O$ open $\Rightarrow\ f^{-1}(O)$ open
(3) $\forall C \subseteq Y\colon\ C$ closed $\Rightarrow\ f^{-1}(C)$ closed

Proof. Assume that f is continuous on X and let $O \subseteq Y$ be open. Then for each $x_0 \in f^{-1}(O)$, there is $\epsilon > 0$ such that $B_Y(f(x_0), \epsilon) \subseteq O$. The continuity of f at x_0 phrased via (24.4) then gives $\delta > 0$ such that $B_X(x_0, \delta) \subseteq f^{-1}(O)$ thus proving that $f^{-1}(O)$ is open.

We have shown (1) $\Rightarrow$ (2). For the converse, assume (2), pick x_0 and, given $\epsilon > 0$, let $O := B_Y(f(x_0), \epsilon)$. Since $x_0 \in f^{-1}(O)$ and O is open, $f^{-1}(O)$ is open and so there exists $\delta > 0$ such that $B_X(x_0, \delta) \subseteq f^{-1}(B_Y(f(x_0), \epsilon))$. But that implies (24.4) proving continuity of f at x_0. As x_0 was arbitrary, we get continuity of f everywhere.

The above proves the equivalence of (1) and (2). As for (3), this is equivalent to (2) by the fact that $f^{-1}(Y \smallsetminus O) = X \smallsetminus f^{-1}(O)$. □

The properties (2) and (3) are phrased using only the notions of open and closed sets and are thus only the property of the *topology* induced by the metric structure of the underlying spaces. This is the basis of:

Definition 24.10 (Continuity in topological spaces) *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. A map $f\colon X \to Y$ with* $\mathrm{Dom}(f) = X$ *is said to be* continuous *if*

$$\forall O \in \mathcal{S}\colon f^{-1}(O) \in \mathcal{T} \tag{24.24}$$

(Here $\mathcal{T}$ is the prescribed collection of open subsets of X and $\mathcal{S}$ is the collection of open subsets of Y, with both subject to the axioms making these a topology.)

While phrasing continuity this way is very elegant, matters get more complicated if we want to talk about continuity at a point. (This is still possible in topological spaces but the definition is more intricate.) The restriction to maps with full domain is essential albeit easily circumvented by resorting to relative topologies. Note that the proof of the Composition Rule becomes very elementary in this context.

24.4 Open and closed maps.

As was just discussed, continuous functions are exactly those that *preimage* open, resp., closed sets into open, resp., closed sets. A natural question is: What are the properties of the functions that *image* open, resp., closed sets into open, resp., closed sets. We give these a formal name in:

Definition 24.11 (Open and closed maps) *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. A function $f\colon X \to Y$ with $\mathrm{Dom}(f) = X$ is then said to be*

(1) open *(or is an* open map*) if*

$$\forall O \subseteq X\colon \; O \text{ open (in } X) \;\Rightarrow\; f(O) \text{ open (in } Y) \tag{24.25}$$

or, formally, $\forall O \in \mathcal{T}\colon f(O) \in \mathcal{S}$.

(2) closed *(or is an* closed map*) if*

$$\forall C \subseteq X\colon \; C \text{ closed (in } X) \;\Rightarrow\; f(C) \text{ closed (in } Y) \tag{24.26}$$

or, formally, $\forall O \in \mathcal{T}\colon Y \smallsetminus f(X \smallsetminus O) \in \mathcal{S}$

We caution that reader that (2) is a distinct property from (1). This is because complementation does not work the same way with images as with preimages with the exception of bijections:

Lemma 24.12 *Let $f\colon X \to Y$ be a bijection. Then f open $\Leftrightarrow$ f closed.*

We leave the proof of this lemma to homework. To give some examples, note that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \frac{x}{1+x^2} \tag{24.27}$$

is closed (because it maps $\mathbb{R}$ onto a compact interval; the Bolzano-Weierstrass Theorem then does the job) but not open (because it maps $\mathbb{R}$, which is open onto a compact and thus closed interval) while the map

$$g(x) := \frac{x|x|}{1+x^2} \tag{24.28}$$

is open (being a continuous bijection of $\mathbb{R}$ onto an open subset $(-1, 1)$ of $\mathbb{R}$) but not closed (because it maps $\mathbb{R}$ which is closed onto an open set $(-1, 1)$).

Closed maps are more prevalent thanks to the fact that the restriction to compact sets is automatic for continuous functions:

Theorem 24.13 (Continuous image of a compact set is compact) *Let $f\colon X \to Y$ be a map between (topological or) metric spaces X and Y with $\mathrm{Dom}(f) = X$. Then*

$$X \text{ compact} \;\wedge\; f \text{ continuous} \;\Rightarrow\; f(X) \text{ compact} \tag{24.29}$$

Proof. Suppose X is compact and let $f\colon X \to Y$ be continuous with $\mathrm{Dom}(f) = X$. Consider an open cover $\{O_\alpha\colon \alpha \in I\}$ of $f(X)$. The continuity of f (along with $\mathrm{Dom}(f) = X$) then implies that $\{f^{-1}(O_\alpha)\colon \alpha \in I\}$ is an open cover of X. Since X is compact, there exists $F \subseteq I$ finite such that $\{f^{-1}(O_\alpha)\colon \alpha \in F\}$ is still an open cover of X. But then $\{O_\alpha\colon \alpha \in F\}$ is a cover of $f(X)$, proving compactness thereof. $\square$

Corollary 24.14 *A continuous function (with full domain) on a compact space is closed.*

Proof. Let $f\colon X \to Y$ be continuous with X compact and $\mathrm{Dom}(f) = X$. Let $C \subseteq X$ be closed. Then C is compact and so $f(C)$ is compact and thus closed. □

The property of being open is harder to achieve which is the reason why open maps appear in important theorems in analysis; for instance, complex analysis (where the Open Mapping Theorem states that each non-constant holomorphic function is open) and functional analysis (where its name sake, a.k.a. the Banach-Schauder Theorem, says that a surjective continuous linear map between Banach spaces are open).

The main reason for our interest in these is:

Lemma 24.15 *Let $f\colon X \to Y$ be bijective and thus invertible. Then (assuming that X and Y are topological or metric spaces),*

$$f \text{ open } \Leftrightarrow f^{-1} \text{ continuous} \tag{24.30}$$

Proof. Pick $U \subseteq X$ and let $O := f(U)$. To ease notation, denote $g := f^{-1}$. Then (with g^{-1} denoting the preimage function) $g^{-1}(U) = O$ and so "$g^{-1}(U)$ open" is equivalent to "$f(U)$ open." By Theorem 24.9, "f open" is thus equivalent to "g continuous." □

As an application we show:

Lemma 24.16 *Let $n \in \mathbb{N}$ obey $n \geqslant 2$. Then $f(x) := x^{1/n}$ with $\mathrm{Dom}(f) := [0,\infty)$ is continuous.*

Proof. f is the inverse of $g(x) := x^n$ (with $\mathrm{Dom}(g) := [0,\infty)$) which, by strict monotonicity and everywhere invertibility, maps intervals in $[0,\infty)$ into intervals preserving the open/closed status of the endpoints. By the characterization of open sets in $\mathbb{R}$ as countable unions of open intervals, every open set in $[0,\infty)$ is mapped onto an open set in $[0,\infty)$ by g. Hence g is open and so, by Lemma 24.15, $f = g^{-1}$ is continuous. □

We finish with a consequence of Theorem 24.13:

Corollary 24.17 *Let X be a metric space and $f\colon X \to \mathbb{R}$ a continuous function with $\mathrm{Dom}(f) = X$. Then for all compact $A \subseteq X$, the image $f(A)$ is bounded and there exist $x_0, x_1 \in A$ such that*

$$f(x_0) = \inf\{f(x)\colon x \in A\} \wedge f(x_1) = \sup\{f(x)\colon x \in A\} \tag{24.31}$$

To put this in words, continuous real-valued functions on a compact set achieve their minimum and maximum.

Proof. Let $A \subseteq X$ be compact. By Theoreem 24.13, $f(A)$ is compact in $\mathbb{R}$. The Heine-Borel Theorem implies that $f(A)$ is bounded and closed. Since the infimum/supremum of f on A are adherent points of $f(A)$, we have $\inf_{x\in A} f(x) \in f(A)$ and $\sup_{x\in A} f(x) \in f(A)$. In particular, there are $x_0, x_1 \in A$ such that (24.31) holds. □

Compactness is of course essential for the statement in Corollary 24.17 to hold. The conclusion is very useful throughout mathematics.