A NEW DESCRIPTIVE DEFINITION OF THE WARD INTEGRAL

R. HENSTOCK*.

In the paper [4], A. J. Ward gives definitions of major and minor functions of a finite function f with respect to another finite function ϕ that is not necessarily of bounded variation, by using increments only, and not derivatives. These definitions lead to an integral

$$(W)\int_a^b f(x)\,d\phi(x) \tag{1}$$

with the properties that if the corresponding elementary Stieltjes integral, or Pollard's modified Stieltjes integral, exists then Ward's integral exists with the same value†. Further, if the corresponding Lebesgue-Stieltjes integral exists, so that ϕ is necessarily of bounded variation then again Ward's integral exists, and a simple relation connects the two. There is equality if ϕ is also continuous‡. If ϕ is continuous and strictly increasing then the Ward integral is equivalent to the Perron-Stieltjes integral§ and to an integral of Denjoy-Stieltjes type||.

Recently, in the course of two papers¶, I gave a descriptive definition of an integral involving convergence factors. On the suggestion of a referee, this paper is devoted to the corresponding descriptive definition of the Ward integral. A simple generalization enables us to obtain the descriptive definition of an integral of an interval function with values in a Banach space.

1. *Definitions.*

Ward's definition of the integral (1) is as follows. Given any functions f, ϕ, defined and finite everywhere in $a \leqslant x \leqslant b$, a finite function F is a *major function of* f, ϕ, *in* $[a, b]$ if $F(a) = 0$, and if for $a \leqslant x \leqslant b$ there is a $\delta(x) > 0$ such that

$$F(t) \geqslant F(x)+f(x)\{\phi(t)-\phi(x)\} \text{ if } 0 \leqslant t-x \leqslant \delta(x); \tag{2}$$

$$F(t) \leqslant F(x)+f(x)\{\phi(t)-\phi(x)\} \text{ if } 0 \geqslant t-x \geqslant -\delta(x). \tag{3}$$

If $x = a$ or $x = b$ we only consider the one appropriate inequality. Further, the finite function F is a *minor function of* f, ϕ, if $-F$ is a major

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† Ward [4], 587.

‡ See, for example, Saks [3], 208, Theorem (8 . 1).

§ Ward [4], 587, proves this result in the case $\phi(x) \equiv x$, but the proof can easily be generalized to the case when ϕ is continuous and strictly increasing. See [2].

|| See [2].

¶ These will now be published as papers [1], [2].

function of $-f$, ϕ. Let $\Phi(b)$ be the greatest lower bound of $F(b)$ for all major functions F, and let $\Psi(b)$ be the least upper bound of $F(b)$ for all minor functions F. If

$$\Phi(b) = \Psi(b)$$

their common value is the definition of the integral (1).

The new descriptive definition of the Ward integral is obtained in the following way. First we say that a family $\mathscr{R}$ of intervals is *right-complete* in $[a, b]$, if to each point x in $a \leqslant x < b$ there is an

$$h_1(\mathscr{R}, x) = h_1(x) > 0$$

such that every interval $(x, x+h)$ in $[a, b]$, with $0 < h \leqslant h_1(x)$, lies in $\mathscr{R}$. A family $\mathscr{L}$ of intervals is *left-complete* in $[a, b]$ if to each point x in $a < x \leqslant b$ there is an

$$h_2(\mathscr{L}, x) = h_2(x) > 0$$

such that every interval $(x-h, x)$ in $[a, b]$, with $0 < h \leqslant h_2(x)$, lies in $\mathscr{L}$.

Given three functions H, f, ϕ, in that order, each defined and finite in $[a, b]$, and a left-complete family $\mathscr{L}$ and a right-complete family $\mathscr{R}$ of intervals in $[a, b]$, we can define a function of intervals,

$$c(H, f, \phi;\ \mathscr{L}, \mathscr{R};\ u, v) = c(u, v),\ \textit{for}\ \ (u, v) \in \mathscr{L} \cup \mathscr{R},$$

such that if (u, v) is in $\mathscr{R}$ but not in $\mathscr{L}$,

$$c(u, v) = |H(v) - H(u) - f(u)\{\phi(v) - \phi(u)\}|;$$

if (u, v) is in $\mathscr{L}$ but not in $\mathscr{R}$,

$$c(u, v) = |H(v) - H(u) - f(v)\{\phi(v) - \phi(u)\}|;$$

and otherwise $c(u, v)$ is the greater of the moduli. Let

$$V \equiv V(H, f, \phi;\ \mathscr{L}, \mathscr{R};\ a, b)$$

be the total variation over $[a, b]$ of the interval function $c(u, v)$, using only the intervals (u, v) for which $c(u, v)$ has been defined, *i.e.*

$$V = \sup \sum_{p=1}^{n} c(x_p, x_{p+1})$$

for all divisions $a = x_1 < x_2 < \ldots < x_{n+1} = b$ for which

$$(x_p, x_{p+1}) \in \mathscr{L} \cup \mathscr{R} \quad (p = 1, \ldots, n).$$

Then we say that the function H is *variationally equivalent* to the functions f, ϕ, in $[a, b]$, if, given $\epsilon > 0$, there are two families $\mathscr{L}_1$, $\mathscr{R}_1$ of intervals depending on ϵ, respectively left-complete and right-complete in $[a, b]$, such that

$$V(H, f, \phi;\ \mathscr{L}_1, \mathscr{R}_1;\ a, b) < \epsilon. \tag{4}$$

An alternative definition is as follows. The function H is *variationally equivalent* to the functions f, ϕ, in $[a, b]$, if, given $\epsilon > 0$, there are two families $\mathscr{L}_1$, $\mathscr{R}_1$, of intervals depending on ϵ, respectively left-complete and right-complete in $[a, b]$, and a monotone increasing function χ such that

$$0 = \chi(a) \leqslant \chi(b) < \epsilon, \tag{5}$$

$$c(u, v) \leqslant \chi(v) - \chi(u) \quad \big((u, v) \in \mathscr{L}_1 \cup \mathscr{R}_1\big). \tag{6}$$

From (6) it follows that

$$V \leqslant \chi(b) - \chi(a),$$

so that (5) then gives (4). Conversely, because of (4) we can construct a function χ with the required properties by putting

$$\chi(x) = V(H, f, \phi;\ \mathscr{L}_1, \mathscr{R}_1;\ a, x) \equiv V(a, x),$$

since $$V(a, x) + V(x, y) \leqslant V(a, y) \text{ for } a \leqslant x < y \leqslant b.$$

Thus the two definitions are completely alternative.

If the function H is variationally equivalent to functions f, ϕ, in $[a, b]$ the difference $H(b) - H(a)$ can be called the *variational integral* of f with respect to ϕ in $[a, b]$.

2. *The equivalence of the variational integral and Ward's integral.*

In this section we see that the variational integral provides a new descriptive definition of the Ward integral.

THEOREM. *If for $a \leqslant x \leqslant b$ there exists*

$$(W) \int_a^x f(x)\, d\phi(x) \equiv I(x) \tag{7}$$

then I is variationally equivalent to f, ϕ in $[a, b]$. Conversely, if H is variationally equivalent to f, ϕ in $[a, b]$ then f is Ward-integrable with respect to ϕ there, with indefinite Ward integral H.

Suppose first that (7) exists for $a \leqslant x \leqslant b$, and let F, G be, respectively, a major and a minor function of f, ϕ such that

$$F(b) - G(b) < \epsilon. \tag{8}$$

From (2, 3) we see that for $0 < t - x \leqslant \delta(x)$, $a \leqslant x < b$, where $\delta(x)$ is the smaller of the two functions for F, G separately,

$$\{F(t) - G(t)\} - \{F(x) - G(x)\} = [F(t) - F(x) - f(x)\{\phi(t) - \phi(x)\}]$$
$$- [G(t) - G(x) - f(x)\{\phi(t) - \phi(x)\}] \geqslant 0.$$

Similarly, for $0 > t - x \geqslant -\delta(x)$, $a < x \leqslant b$,

$$\{F(t) - G(t)\} - \{F(x) - G(x)\} \leqslant 0.$$

Thus $F - G$ is monotone increasing. If we let $G(x)$, $G(t)$ tend respectively to $I(x)$, $I(t)$ we see that $F - I$ is monotone increasing. Similarly for $I - G$.

Again, for $0 < t - x \leqslant \delta(x)$, $a \leqslant x < b$,

$$\begin{aligned} I(t) - I(x) &= \{F(t) - F(x)\} - \{F(t) - I(t)\} + \{F(x) - I(x)\} \\ &\geqslant f(x)\{\phi(t) - \phi(x)\} - \{F(t) - G(t)\} + \{F(x) - G(x)\}, \\ I(t) - I(x) &= \{G(t) - G(x)\} + \{I(t) - G(t)\} - \{I(x) - G(x)\} \\ &\leqslant f(x)\{\phi(t) - \phi(x)\} + \{F(t) - G(t)\} - \{F(x) - G(x)\}. \end{aligned}$$

Similar inequalities hold for $-\delta(x) \leqslant t-x < 0$, $a < x \leqslant b$.

Thus the function

$$\chi = F - G$$

is monotone increasing and satisfies (5), by (8), and satisfies (6) with H replaced by I. Thus $I(b)$ is the variational integral of f with respect to ϕ over $[a, b]$.

Conversely, let us suppose that H is variationally equivalent to f, ϕ in $[a, b]$, with $H(a) = 0$. Then from (5, 6), $H+\chi$ is a major and $H-\chi$ a minor function of f, ϕ in $[a, b]$, and

$$0 \leqslant \{H(b)+\chi(b)\}-\{H(b)-\chi(b)\} = 2\chi(b) < 2\epsilon.$$

As $\epsilon > 0$ is arbitrary, the Ward integral (1) exists equal to $H(b)$.

3. *Generalizations.*

If f, ϕ are functions whose values are complex numbers we can define directly the variational integral of f with respect to ϕ in $[a, b]$, without having to separate f, ϕ into their real and imaginary parts. More generally, if there is a norm on the field of values of

$$H(t)-H(x)-f(x)\{\phi(t)-\phi(x)\} \quad (x \neq t) \tag{9}$$

for x, t in the interval of integration we can define directly the variational integral, simply by replacing the modulus, that occurs in the definition of $c(u, v)$, by the norm. For example, if the values of f lie in a Banach space, and if ϕ is real or complex-valued, we take the values of H to lie in the same Banach space, and then (9) also lies in the space. A suitable norm will then be the norm of the Banach space.

The theory can be extended to interval functions. The function of intervals (u, v) used in Riemann-Stieltjes integration,

$$f(\xi)\{\phi(v)-\phi(u)\} \quad (u \leqslant \xi \leqslant v), \tag{10}$$

is many-valued, the value depending on the parameter ξ. From this it is usual to construct two single-valued functions of intervals,

$$\sup_{u \leqslant \xi \leqslant v} f(\xi)\{\phi(v)-\phi(u)\}, \quad \inf_{u \leqslant \xi \leqslant v} f(\xi)\{\phi(v)-\phi(u)\}$$

to obtain an integration of Darboux type. However, we can put

$$f(\xi)\{\phi(v)-\phi(u)\} = f(\xi)\{\phi(\xi)-\phi(u)\}+f(\xi)\{\phi(v)-\phi(\xi)\}$$

and obtain (depending on the position of ξ) one or two single-valued functions of intervals that are used in the definition of Ward integration and variational equivalence. Thus in general we need two interval functions, Y_l defined to be "left-hand", and Y_r to be "right-hand". Using these we can define the variational integral of Y_l, Y_r. If $\mathscr{L}_2$, $\mathscr{R}_2$

are families of intervals in $[a, b]$, respectively left-complete and right-complete, and if Y_l is defined in $\mathscr{L}_2$, Y_r in $\mathscr{R}_2$, with values real or complex or in a Banach space, and normed by

$$\|\cdots\|,$$

then we can define, for the values of H in the same space,

$$c(H, Y_l, Y_r; \mathscr{L}_2, \mathscr{R}_2; u, v) = c(u, v) \quad \big((u, v) \in \mathscr{L}_2 \cup \mathscr{R}_2\big),$$

such that if (u, v) is in $\mathscr{R}_2$ but not in $\mathscr{L}_2$,

$$c(u, v) = \|H(v) - H(u) - Y_r(u, v)\|;$$

if (u, v) is in $\mathscr{L}_2$ but not in $\mathscr{R}_2$,

$$c(u, v) = \|H(v) - H(u) - Y_l(u, v)\|;$$

and otherwise $c(u, v)$ is the greater of the norms.

The rest of the definition of the variational integral of Y_l, Y_r is as before. Taking $Y_l(u, v) = Y_r(u, v)$ this integral then includes the Burkill integral of Y_l, as can easily be seen.

It is easy to free the variational integral from its dependence on the interval $a \leqslant x \leqslant b$ by observing that $\mathscr{L}$ and $\mathscr{R}$ are, respectively, sets of left-hand and right-hand neighbourhoods of the points x in $a \leqslant x \leqslant b$. The union of the intervals of $\mathscr{L}$ includes $(a, b]$, and can be arranged to include a as well; and a similar result holds for $\mathscr{R}$.

More generally, we integrate over some space E of points. Instead of families $\mathscr{L}$, $\mathscr{R}$, we have families $\mathscr{F}$ of sets S of E, such that the union of the S in each $\mathscr{F}$ is E. For S in $\mathscr{F}$ there is defined a set-function $Y(S)$ that may be many-valued (to correspond with Y_l, Y_r) with values in a space with norm $\|\ldots\|$. If $H(S)$ is a set-function in E we write

$$c(S) = \|H(S) - Y(S)\|, \tag{11}$$

where the convention is that if Y is many-valued we use the value that makes the norm the greatest. Or we may have to take the supremum if there is an infinity of values. Then

$$V(H, Y; \mathscr{F}; E) = \sup \Sigma\, c(S),$$

the sum being taken over any collection of disjoint sets S. The rest of the definition of variational equivalence goes through as before. There arises a problem of uniqueness. If two set functions are additive and are variationally equivalent to Y, are they equal ? A solution to this problem will depend on the norm in the space of values and on the families $\mathscr{F}$, and it seems vital that we should not leave any part of E uncovered by the families $\mathscr{F}$.

This form of the variational integral can handle (10) as it is, and will probably include the abstract Lebesgue integral, and so the Banach-

Dunford-Pettis integral of functions with values in a Banach space. If the values lie in a metric space with metric

$$M(H, Y)$$

then we would replace (11) by

$$c(S) = M\big(H(S),\ Y(S)\big).$$

But if there is no operation of addition it would be difficult to say which function H variationally equivalent to Y was to be the variational integral; we would have to define the integral to be the equivalence class of functions H variationally equivalent to Y, and then the whole significance of integration would be lost.

If additivity is not required, there is one final generalization. We are given a non-empty family $\mathscr{I}$ of sets S, a space $\mathscr{G}$ containing neighbourhoods N of a point p, a space $\mathscr{H}$, and a family of mappings sending $\mathscr{G}$ into $\mathscr{H}$, with the property that, given x in $\mathscr{G}$ and y in $\mathscr{H}$ there is a mapping $\mathscr{M}(x, y)$ of the family that takes x into y.

A mapping $\chi(S)$ of the sets of a non-empty subfamily $\mathscr{J}$ of $\mathscr{I}$, into sets of $\mathscr{G}$, is called *monotone* if $\chi(S)$, $\chi(S')$ are disjoint when S, S' are disjoint. The mapping is *bounded by a set N of $\mathscr{G}$* if $\chi(S)$ is contained in N for all S of $\mathscr{J}$.

Functions $G(S)$ (possibly many-valued) and $H(S)$ (single-valued) of sets S in $\mathscr{I}$, with values in $\mathscr{H}$, are *variationally equivalent* ($\mathscr{G}$, $\mathscr{H}$, $\mathscr{I}$, N) if, given a neighbourhood N of p, there are a non-empty subfamily $\mathscr{J}$ contained in $\mathscr{I}$, and a monotone mapping χ bounded by N, such that for each S in $\mathscr{J}$, if y has any value of $G(S)$, there is an x in $\chi(S)$ such that $H(S)$ is in $\mathscr{M}(x, y)\chi(S)$.

This section has been written to show the versatility of the definition of variational equivalence, and to give pointers to further research. The principal kinds of integrals not covered by the ideas of this section are those of general Denjoy, Cesàro-Perron or Abel-Perron type. In the paper [**2**] I will give a generalization of $c(u, v)$ to cover integrals equivalent to the last two. An integral equivalent to the general Denjoy integral can be defined as a variational integral by modifying the definitions of the left-complete and right-complete families of intervals. I hope to deal with this in a paper "N-variation and N-variational integrals".

References.

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The University,
Bristol