HW#9: due Fri 6/9/2023, 11:59PM

This exercise practices (real) analytic functions and Fourier series.

Problem 1: Let $f\colon \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ be such that f is analytic at x_0. This means that there exists $r \in (0, \infty]$ and a sequence $\{c_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with

$$\limsup_{n\to\infty} |c_n|^{1/n} \leqslant \frac{1}{r}$$

and

$$\forall x \in (x_0 - r, x_0 + r)\colon\ \ x \in \mathrm{Dom}(f) \ \wedge\ f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

Prove that f is analytic at each $a \in (x_0 - r, x_0 + r)$. (This requires showing that f can be expressed as a power series in $x - a$ for each $a \in (x_0 - r, x_0 + r)$.)

Problem 2: Let $f\colon \mathbb{R} \to \mathbb{R}$ be analytic at x_0. Prove that

$$\exists A \in (0, \infty)\, \exists \delta > 0\, \forall n \in \mathbb{N}\colon \quad \sup_{x\in(x_0-\delta, x_0+\delta)} \left| f^{(n)}(x) \right| \leqslant A\delta^{-n} n!$$

(Taylor's Theorem then shows that this is also sufficient for analyticity at x_0.)

Problem 3: Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be functions and let $x_0 \in \mathbb{R}$ be such that g is analytic at x_0 and f is analytic at $g(x_0)$. Express $f \circ g$ as a power series centered at x_0 and thus show that $f \circ g$ is analytic at x_0.
Note: You can use this (and facts about exp and log we proved in class) to prove that power function, $f(x) := x^\alpha$, is analytic on $\mathrm{Dom}(f) := (0, \infty)$ for each $\alpha \in \mathbb{R}$.

Problem 4: Recall that the tangent function is defined by $\tan(x) := \frac{\sin(x)}{\cos(x)}$. We take $\mathrm{Dom}(\tan) := (-\pi/2, \pi/2)$. Prove the following:

(1) tan is continuous and strictly increasing on its domain with $\mathrm{Ran}(\tan) = \mathbb{R}$
(2) the inverse of tan, often called arctan, is differentiable on $\mathbb{R}$ with

$$\forall x \in \mathbb{R}\colon\ \ \arctan'(x) = \frac{1}{1 + x^2}$$

(3) arctan is analytic on $(-1, 1)$ with

$$\forall x \in (-1, 1)\colon\ \ \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

(4) The series converges at $x = 1$ and we have

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Note: Part (4) requires exchange of the limits $x \to 1^-$ and the infinite sum which is often referred to Abel's Theorem that deals with limits of power series at their convergence radius in large generality. In this particular case, you can control this exchange directly.

Problem 5: Prove that the linear span (over $\mathbb{C}$) of functions $\{z \mapsto z^n \colon n \in \mathbb{Z}\}$ is an algebra that satisfies the conditions of the complex-valued Stone-Weierstrass Theorem on the metric space $\{z \in \mathbb{C} \colon |z| = 1\}$ endowed with the Euclidean metric on $\mathbb{C}$. Use this to prove the following:

(1) For each $f \colon [0,1] \to \mathbb{R}$ continuous with $f(0) = f(1)$ and each $\epsilon > 0$, there is $N \in \mathbb{N}$ and coefficients $c_{-N}, \dots, c_N \in \mathbb{C}$ such that

$$\sup_{x \in [0,1]} \left| f(x) - \sum_{n=-N}^{N} c_n \mathrm{e}^{2\pi \mathrm{i} n x} \right| < \epsilon$$

(2) Assuming that f is as in (1), we then have

$$\forall x \in [0,1]\ \forall \alpha \in [0,1] \smallsetminus \mathbb{Q} \colon \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x + k\alpha \bmod 1) = \int_0^1 f(z)\mathrm{d}z$$

(3) Give an example of an f such that the statement in (2) is FALSE for $\alpha \in [0,1] \cap \mathbb{Q}$.

Note: What (2) shows is that irrational rotations of the circle become asymptotically uniformly distributed over $[0,1]$. This gives a basic justification of the use of quasirandom numbers instead of truly random numbers in computer simulations.

Problem 6: Let $f \in C([0,1], \mathbb{R})$ be such that $f(0) = f(1) = 0$ and for each $n \in \mathbb{N}$ let

$$c_n := 2 \int_0^1 f(x) \sin(\pi n x) \mathrm{d}x$$

Prove that

$$\sum_{k=1}^{\infty} |c_k| < \infty \quad \Rightarrow \quad \sum_{k=1}^{n} c_k \sin(\pi k x) \underset{n \to \infty}{\longrightarrow} f(x) \text{ uniformly}$$

Hint: Either follow the corresponding proof for the Fourier series or prove that the linear span of $\{x \mapsto \sin(\pi n x) \colon n \geqslant 1\}$ is dense in $\{f \in C([0,1], \mathbb{R}) \colon f(0) = f(1) = 0\}$ with respect to the uniform metric and argue from there.

Problem 7: (RUDIN) EX 12, PAGE 198

Problem 8: (RUDIN) EX 13, PAGE 198

Problem 9: (RUDIN) EX 14, PAGE 199