HW#8: due Wed 5/31/2022, 11:59PM

This exercise practices pointwise/uniform convergence and equicontinuity. $C(X,Y)$ denotes to the space of continuous functions $X \to Y$ and $C_b(X,Y)$ denotes the set of bounded continuous functions $X \to Y$ endowed with the supremum metric $\rho_\infty(f,g) := \sup_{x\in X} \rho_Y(f(x),g(x))$. We also practice the Weierstrass Approximation Theorem and the Stone-Weierstrass Theorem.

Problem 1: (Exchange of limits) Let $\{a_{m,n}\}_{m\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ be a two-dimensional array such that the row and column limits exist:

$$\forall m \in \mathbb{N}\colon\ b_m := \lim_{n\to\infty} a_{m,n} \text{ exists} \tag{$\star$}$$

and

$$\forall n \in \mathbb{N}\colon\ c_n := \lim_{m\to\infty} a_{m,n} \text{ exists}$$

Assume, in addition to ($\star$), that

$$\lim_{n\to\infty} \sup_{m\in\mathbb{N}} |b_m - a_{m,n}| = 0$$

Prove that then

$$\lim_{m\to\infty} b_m \text{ exists} \ \wedge\ \lim_{n\to\infty} c_n \text{ exists} \ \wedge\ \lim_{m\to\infty} b_m = \lim_{n\to\infty} c_n$$

The existence of all limits is in $\mathbb{R}$ but the supremum is in extended reals.

Problem 2: Prove the following version of a theorem due to Arzelà and Ascoli we stated in class: Let X and Y be metric spaces and $\{f_n\}_{n\in\mathbb{N}}$ and f functions $X \to Y$ such that

$$\{f_n\}_{n\in\mathbb{N}} \text{ uniformly equicontinuous } \ \wedge\ f_n \to f \text{ pointwise}$$

Assume also that X is totally bounded. Then $f_n \to f$ uniformly.

Problem 3: (RUDIN) EX 13, PAGE 167 (Helly selection theorem)

Problem 4: Prove one direction of the functional form of the Arzelà-Ascoli Theorem: Let X and Y be metric spaces with X compact and Y locally compact. Then

$$\forall F \subseteq C_b(X,Y)\colon\quad F \text{ compact } \Rightarrow\ F \text{ closed } \wedge \text{ bounded } \wedge \text{ equicontinuous}$$

Note: The terms "compact, closed, bounded" refer to the metric structure on $C_b(X,Y)$. A metric space is locally compact if closed balls therein are compact.

Problem 5: To demonstrate the usefulness of the space of continuous functions, prove that each metric space (X,ρ_X) admits a completion using the following argument: Fix an element $z_0 \in X$ and, for each $z \in X$, define $f_z\colon X \to \mathbb{R}$ by

$$f_z(x) := \rho_X(z,x) - \rho_X(z_0,x)$$

Then prove

$$\forall z \in X\colon\ f_z \in C_b(X,\mathbb{R})$$

and

$$\forall z,\tilde{z} \in X\colon\ \sup_{x\in X}\big|f_z(x) - f_{\tilde{z}}(x)\big| = \rho_X(z,\tilde{z})$$

Conclude that $\phi\colon X \to C_b(X,\mathbb{R})$ defined by $\phi(z) := f_z$ is an isometry of X onto the subspace $\mathrm{Ran}(\phi)$ of $C_b(X,\mathbb{R})$. Writing $\overline{\mathrm{Ran}(\phi)}$ for the closure of $\mathrm{Ran}(\phi)$ in $C_b(X,\mathbb{R})$, prove that $(\overline{\mathrm{Ran}(\phi)}, \rho_\infty)$ is a completion of (X,ρ).

Problem 6: (Retry of an example from 131AH) Prove that, in any metric space X, a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges if and only if there exists $x \in X$ such that every subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ contains a subsubsequence $\{x_{n_{k_\ell}}\}_{\ell\in\mathbb{N}}$ with $x_{n_{k_\ell}} \to x$. Then use this observation to prove that pointwise convergence of functions $\mathbb{R} \to \mathbb{R}$ (or even $A \to \{0,1\}$ for A the power of continuum) is not metrizable.

Hint: Consider the sequence $\{1_{I_n}\}_{n\in\mathbb{N}}$ for $\{I_n\}_{n\in\mathbb{N}}$ "wandering" subintervals of $[0,1]$.

Note: The pointwise convergence can be cast using a topology, but (as the above shows) not necessarily one that is first countable (which is necessary for metrizibility).

Problem 7: (RUDIN) EX 20, PAGE 169 (Orthogonality to all mononomials)

Problem 8: (Another selection theorem) Let $a < b$ be reals and $\{f_n\}_{n\in\mathbb{N}}$ a uniformly bounded sequence of continuous functions $f_n\colon [a,b] \to \mathbb{R}$. Prove that there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of naturals such that

$$\forall g \in C([a,b])\colon \quad \lim_{k\to\infty} \int_a^b g(t) f_{n_k}(t)\,\mathrm{d}t \quad \text{exists}$$

Note that equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$ is NOT assumed. An interesting follow-up question is whether one can always find an $f\colon [a,b] \to \mathbb{R}$ such that the limit equals $\int_a^b g(t)f(t)\mathrm{d}t$ for each $g \in C([a,b])$. If so, then we say that $f_{n_k} \to f$ *weakly in* L^2.

Problem 9: Prove that the linear vector space

$$\left\{ \sum_{k=1}^{n} \Big(a_k \sin(2\pi k x) + b_k \cos(2\pi k x) \Big) \colon n \in \mathbb{N} \,\wedge\, a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{R} \right\}$$

is dense in the subspace

$$\left\{ f \in C\big([0,1],\mathbb{R}\big) \colon f(0) = f(1) \,\wedge\, \int_0^1 f(x)\,\mathrm{d}x = 0 \right\}$$

of $C\big([0,1],\mathbb{R}\big)$ endowed with the supremum metric.

Problem 10: (RUDIN) EX 21, PAGE 169 (Stone-Weierstrass non-example)