HW#7: due Mon 5/22/2023

This exercise practices uniform convergence and some basic transcendental functions.

Problem 1: Let (X, ϱ_X) and (Y, ϱ_Y) metric spaces and, for each $n \geqslant 1$, let f_n be a continuous function $f_n \colon X \to Y$. Let $f \colon X \to Y$ be a function. Prove that

$$f_n \to f \text{ uniformly } \;\Rightarrow\; \Big(\forall \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}\, \forall x \in X \colon\; x_n \to x \;\Rightarrow\; f_n(x_n) \to f(x)\Big) \qquad (\star)$$

Then give an example for which $f_n \to f$ pointwise and yet the clause on the right of $(\star)$ is FALSE. Finally, prove that for X compact and f continuous, $\Leftarrow$ holds in $(\star)$ as well (under the assumptions of f_n made earlier).

Problem 2: (RUDIN) EX 2-3, PAGE 165 (Sum and product of uniformly convergent sequences of functions.)

Problem 3: (RUDIN) EX 6, PAGE 166 (A series that converges uniformly everywhere but absolutely nowhere.)

Problem 4: (RUDIN) EX 7, PAGE 166 (Convergence of derivatives and uniformity.)

Problem 5: (RUDIN) EX 8, PAGE 166 (A series with prescribed discontinuities.)

Problem 6: (RUDIN) EX 10, PAGE 167 (A series with countably many discontinuities.)

Problem 7: (RUDIN) EX 11, PAGE 167 (Uniform convergence of $\sum_n f_n g_n$ under conditions on f_n and g_n.)

Problem 8: (RUDIN) EX 12, PAGE 167 (A weak form of Dominated Convergence.)

Problem 9: For each $x \in \mathbb{R}$, let

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

where $n!$ is defined recursively by $0! := 1$ and $(n+1)! := (n+1) \cdot n!$. Prove that

$$\forall x, y \in \mathbb{R} \colon\quad \exp(x+y) = \exp(x) \cdot \exp(y)$$

Design two proofs; one by rearranging the product of two infinite series (it suffices to check explicitly the conditions of a theorem that makes this permissible) and another by calculus. Then show

$$\exists \mathrm{e} \in (1, \infty)\, \forall x \in \mathbb{R} \colon\quad \exp(x) = \mathrm{e}^x$$

Here we use the definition of a^x as a continuous extension of a^x defined for $x \in \mathbb{Q}$ by powers and roots.

Problem 10: Define the (standard trig) functions sin and cos by the infinite series

$$\sin(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \;\wedge\; \cos(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Prove the following facts using only these definitions and facts we proved about such convergent power series and differentiable functions in general:

(1) $\forall x \in \mathbb{R}\colon\; \sin(x)^2 + \cos(x)^2 = 1$ and so sin and cos take values in $[-1, 1]$,

(2) the addition formulas hold:

$$\forall x, y \in \mathbb{R}\colon \quad \begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \end{aligned}$$

(3) The number $\pi := 2\inf\{t \geqslant 0\colon\; \cos(t) = 0\}$ obeys $\pi \in (0, \infty)$ and

$$\forall x \in \mathbb{R}\colon \quad \sin(x) = -\cos(x + \pi/2) = -\sin(x - \pi)$$

thus showing that sin and cos are 2π-periodic functions.

Note: (3) gives you a way to define $\pi = 3.1415926\ldots$ in analysis. The relation to the geometry of Euclidean circles is derived from the above facts.
