HW#6: due Mon 5/15/2023

This exercise is largely focused on the Stieltjes integral $\int_a^b f\mathrm{d}g$ defined as the limit of the Riemann-Stieltjes sums $S(f,\mathrm{d}g,\Pi)$ for marked partitions Π of $[a,b]$ as the mesh $\|\Pi\|$ tends to zero. We write

$$\mathrm{RS}(g,[a,b]) = \Big\{f\colon [a,b]\to\mathbb{R}:\ \int_a^b f\mathrm{d}g \text{ exists}\Big\}$$

The definition of $\int_a^b f\mathrm{d}g$ in the textbook goes via upper and lower Darboux sums but that makes it limited to g monotone or, by Jordan decomposition, of bounded variation.

Problem 1: Define $f\colon (0,\infty)\to\mathbb{R}$ by

$$f(x) := \int_1^x \frac{1}{t}\mathrm{d}t$$

Prove the following facts:

(1) f is continuous and strictly increasing on $(0,\infty)$
(2) $\forall x,y\in(0,\infty)\colon f(x\cdot y) = f(x)+f(y)$
(3) f^{-1} exists with $\mathrm{Dom}(f^{-1})=\mathbb{R}$ and obeys $\forall x,y\in\mathbb{R}\colon f^{-1}(x+y) = f^{-1}(x)\cdot f^{-1}(y)$
(4) f^{-1} is continuous on $\mathbb{R}$ and $\exists a>1\,\forall x\in\mathbb{R}\colon f^{-1}(x)=a^x$

Note: This shows that f^{-1} is an exponential function and f is a logarithm.

Problem 2: Let $f,g\colon [a,b]\to\mathbb{R}$ be such that $f\in\mathrm{RS}(g,[a,b])$. Prove that there exists a partition $\Pi=\{t_i\}_{i=0}^n$ of $[a,b]$ such that

$$\forall i=0,\dots,n\colon \sup_{x\in[t_{i-1},t_i]} \big|f(x)\big| <\infty \ \vee\ \big(\forall x,y\in[t_{i-1},t_i]\colon\ g(x)=g(y)\big)$$

In particular, show that if g is NOT constant on any non-degenerate subinterval of $[a,b]$, then $f\in\mathrm{RS}(g,[a,b])$ implies that f is bounded.

Problem 3: Let $a<b$ be reals and $\{\alpha_n\}_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ a sequence such that $\sum_{n\in\mathbb{N}}|\alpha_n|<\infty$. Given a sequence $\{x_n\}_{n\in\mathbb{N}}\in[a,b]^{\mathbb{N}}$, define $h\colon [a,b]\to\mathbb{R}$ by

$$h(x) := \sum_{n=0}^{\infty}\alpha_n 1_{[x_n,\infty)}(x)$$

where we recall that $1_A(x)$ equals one if $x\in A$ and zero otherwise. (The series converges for each x by our assumptions on $\{\alpha_n\}_{n\in\mathbb{N}}$.) Prove that for all continuous $f\colon [a,b]\to\mathbb{R}$,

$$f\in\mathrm{RS}(h,[a,b]) \ \wedge\ \int_a^b f\mathrm{d}h = \sum_{n=0}^{\infty}\alpha_n f(x_n)$$

Then do the same assuming only that f is bounded and continuous at x_n, for all $n\in\mathbb{N}$.
Note: This shows that the Stieltjes integral includes finite sums and convergent series.

Problem 4: Let $a < b$ be reals and let $g\colon [a,b] \to \mathbb{R}$ be right-continuous and of bounded variation; i.e., $V(g,[a,b]) < \infty$. Prove that there exist $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $\sum_{n=0}^{n} |\alpha_n| < \infty$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \in [a,b]^{\mathbb{N}}$ such that for h as in the previous problem,

$$g - h \text{ is continuous } \wedge \ V\big(g-h,[a,b]\big) < \infty$$

Abbreviating $\tilde{g} := g - h$, prove that then for each $f \in \mathrm{RS}(g,[a,b])$,

$$f \in \mathrm{RS}\big(\tilde{g},[a,b]\big) \ \wedge \ \int_a^b f \mathrm{d}g = \sum_{n=0}^{\infty} \alpha_n f(x_n) + \int_a^b f \mathrm{d}\tilde{g}$$

Note: Writing $g = h + \tilde{g}$ as above is also referred to as Jordan decomposition. Such a decomposition is unique; h is then called the *jump part* of g while $\tilde{g}$ is the *continuous part* of g. (In probabilistic applications, this gives a decomposition into a discrete and continuous random variable.) The example $g := 1_{\mathbb{Q}}$ shows that no such decomposition may exist once g is not of bounded variation.

Problem 5: Let $a < b$ be reals and let $f, g\colon [a,b] \to \mathbb{R}$ be functions such that

(1) f is Riemann integrable on $[a,b]$, and
(2) g is continuous on $[a,b]$, differentiable on (a,b) with g' Riemann integrable on $[a,b]$.

Prove that $f \in \mathrm{RS}(g,[a,b])$ and

$$\int_a^b f \mathrm{d}g = \int_a^b f(x) g'(x) \mathrm{d}x$$

Then show that also $g \in \mathrm{RS}(f,[a,b])$ and

$$\int_a^b g \,\mathrm{d}f = f(b)g(b) - f(a)g(a) - \int_a^b f(x) g'(x) \mathrm{d}x$$

Note: The existence and Riemann integrability of g' is crucial here as the integrals $\int_a^b f \mathrm{d}g$ and $\int_a^b g \mathrm{d}f$ may fail to exist when g is only continuous!

Problem 6: Prove the following Mean-Value Theorems: Let $f, g\colon [a,b] \to \mathbb{R}$ be such that f is continuous and g non-decreasing. Then $f \in \mathrm{RS}(g,[a,b])$ and

$$\exists c \in [a,b]\colon \quad \int_a^b f \mathrm{d}g = f(c)\big[g(b) - g(a)\big]$$

Assuming only that f is Riemann integrable and g is non-decreasing, prove that then

$$\exists c \in [a,b]\colon \quad \int_a^b f(x) g(x) \mathrm{d}x = g(a) \int_a^c f(x) \mathrm{d}x + g(b) \int_c^b f(x) \mathrm{d}x$$

Hint: The first statement relies on the Intermediate Value Theorem. For the second statement, write the Riemann integral on the left as $\int_a^b g \mathrm{d}h$ for a suitable h.

Problem 7: Let $f, g\colon [a,b] \to \mathbb{R}$ be functions and assume that f is bounded and g is continuous and of bounded variation; i.e., $V(g,[a,b]) < \infty$. Let $v_g\colon [a,b] \to \mathbb{R}$ be defined by $v_g(t) := V(g,[a,t])$. Prove that

$$f \in \mathrm{RS}\big(g,[a,b]\big) \;\Leftrightarrow\; f \in \mathrm{RS}\big(v_g,[a,b]\big)$$

and, if both TRUE, then also $|f| \in \mathrm{RS}(v_g,[a,b])$ and

$$\left|\int_a^b f\mathrm{d}g\right| \leqslant \int_a^b |f|\mathrm{d}v_g$$

Hint: For the implication $\Rightarrow$ above, consider first showing that for each $\epsilon > 0$ there is $\delta > 0$ such that if $\Pi = \{t_i\}_{i=0}^n$ is a partition of $[a,b]$, then

$$\|\Pi\| < \delta \;\Rightarrow\; \sum_{i=1}^n \Big| v_g(t_i) - v_g(t_{i-1}) - \big|g(t_i) - g(t_{i-1})\big| \Big| < \epsilon$$

The implication $\Rightarrow$ holds without a continuity assumption on g but that requires treating discontinuity points of g explicitly.

Problem 8: Prove Cousin's Theorem: Let $a < b$ be reals and assume that $\mathcal{I}$ is a collection of non-degenerate closed subintervals of $[a,b]$ with the following property: For each $x \in [a,b]$ there is $\delta > 0$ such that all non-degenerate closed intervals $[c,d]$ satisfying

$$[c,d] \subseteq [a,b] \;\wedge\; x \in [c,d] \;\wedge\; d - c < \delta$$

belong to $\mathcal{I}$. Prove that then there is a partition of $[a,b]$ consisting only of intervals in $\mathcal{I}$, i.e., that there are $a = t_0 < t_1 < \cdots < t_n = b$ satisfying

$$\forall i = 1,\dots,n\colon\; [t_{i-1},t_i] \in \mathcal{I}$$

Note: Cousin's theorem ensures that, in the definition of Henstock-Kurzweil integral, for each gauge function there is at least one marked partition obeying the gauge restriction.
