HW#5: due Mon 5/8/2023

This exercise practices the Riemann/Darboux integration theory (which we showed to be equivalent). In the exercises where you are asked to prove Riemann integrability, do not just call on Lebesgue's characterization of Riemann integrability. Rather, find a simpler argument that does not use zero-length property.

Problem 1: Let $h\colon [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and let $F\colon \mathbb{R} \to \mathbb{R}$ be bounded and continuous on $\text{Dom}(F)$ which we assume obeys $\overline{\text{Ran}(h)} \subseteq \text{Dom}(F)$. Prove that $F \circ h$ is Riemann integrable on $[a,b]$. (This implies that if f is Riemann integrable, then so is f^2 and, if $f \geqslant 0$, also $\sqrt{f}$, and similarly for other basic functions.)

Problem 2: Let $f\colon [a,b] \to \mathbb{R}$ be bounded and such that

$$\forall x \in (a,b)\colon \lim_{z \to x} f(z) \text{ exists}$$

(Note that this says nothing about continuity of f.) Prove that f is Riemann integrable.

Problem 3: (RUDIN) EX 2, PAGE 138 (Vanishing integral implies vanishing continuous integrand.) Then define a function $f\colon [0,1] \to [0,1]$ such that

$$\big\{x \in [0,1]\colon f \text{ NOT continuous at } x\big\} \text{ is uncountable and dense in } [0,1]$$

and yet f is Riemann integrable and $\int_0^1 f(x)\mathrm{d}x = 0$.

Problem 4: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded function (with $\text{Dom}(f) = \mathbb{R}$). Define functions $M_f, m_f\colon \mathbb{R} \to \mathbb{R}$ by

$$M_f(x) := \inf_{\delta > 0} \sup_{z\colon |z-x|<\delta} f(z) \quad \text{and} \quad m_f(x) := \sup_{\delta > 0} \inf_{z\colon |z-x|<\delta} f(z)$$

and, given $s > 0$, let

$$U_s := \big\{x \in \mathbb{R}\colon M_f(x) - m_f(x) < s\big\}$$

Prove that the following holds for all $s > 0$:

(1) U_s is open
(2) if f is Riemann integrable on $[a,b]$, then U_s is dense in $[a,b]$.

Now use is to conclude that, if $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then

$$\big\{x \in [a,b]\colon f \text{ continuous at } x\big\} \text{ is dense in } [a,b]$$

Problem 5: Let $a < b$ be reals and $f, g\colon [a,b] \to \mathbb{R}$ bounded functions. Prove that the upper Darboux integral is subadditivive,

$$\overline{\int_a^b} (f+g)(x)\mathrm{d}x \leqslant \overline{\int_a^b} f(x)\mathrm{d}x + \overline{\int_a^b} g(x)\mathrm{d}x$$

and the lower Darboux integral is superadditive,

$$\underline{\int_a^b}(f+g)(x)\mathrm{d}x \geqslant \underline{\int_a^b} f(x)\mathrm{d}x + \underline{\int_a^b} g(x)\mathrm{d}x$$

Use these to prove that, if f and g are both Darboux/Riemann integrable, then so is $f+g$ and the integral is additive. Then give explicit examples of f and g for which the above inequalities are strict.

Problem 6: (RUDIN) EX 8, PAGE 138 (Improper integral and integral test)

Problem 7: For each $x > 1$ define $\log(x) := \int_1^x t^{-1}\mathrm{d}t$. Use ideas from previous exercise to prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right)$$

exists in $\mathbb{R}$.

Problem 8: (RUDIN) EX 10, PAGE 139 (Hölder's inequality) Solve first for the Riemann integral (instead of the Stieltjes integral). You may want to start by proving that if $f\colon [a,b] \to \mathbb{R}$ is integrable, then

$$f \geqslant 0 \;\Rightarrow\; \int_a^b f(x)\mathrm{d}x \geqslant 0$$

and using this to show the *Cauchy-Schwarz inequality*: For all $f, g\colon [a,b] \to \mathbb{R}$ that are Riemann integrable,

$$\left|\int_a^b f(x)g(x)\mathrm{d}x\right| \leqslant \left(\int_a^b f(x)^2\mathrm{d}x\right)^{1/2}\left(\int_a^b g(x)^2\mathrm{d}x\right)^{1/2}$$

Problem 9: Let f be Riemann integrable on $[a,b]$ and set $F(x) := \int_a^x f(t)\mathrm{d}t$. Prove that

$$\forall x \in (a,b)\colon \quad \lim_{z\to x} f(z) \text{ exists } \Rightarrow\; F'(x) \text{ exists } \wedge\; F'(x) = \lim_{z\to x} f(z)$$

Then generalize this to one-sided limits as

$$\forall x \in [a,b)\colon \quad f(x^+) \text{ exist } \Rightarrow\; \frac{\mathrm{d}F}{\mathrm{d}x^+}(x) \text{ exists } \wedge\; \frac{\mathrm{d}F}{\mathrm{d}x^+}(x) = f(x^+)$$

Finally, given an example of a function f such that (for some $x \in (a,b)$) $F'(x)$ exists yet $\lim_{z\to x} f(z)$ does NOT.

Problem 10: (RUDIN) EX 15, PAGE 141 (Bounding f' by f and f''.)