HW#4: due Mon 5/1/2023

This exercise practices derivative, Mean-Value Theorem and Taylor's Theorem and related topics from differential calculus. The last problem uses the Riemann integral.

Problem 1: Prove the Inverse Function Rule in the following form: Let $f\colon \mathbb{R} \to \mathbb{R}$ be injective on its domain and let $x \in \operatorname{int}(\operatorname{Dom}(f))$ be such that $f(x) \in \operatorname{int}(\operatorname{Ran}(f))$. Assume that the inverse function f^{-1} of f (with $\operatorname{Dom}(f^{-1}) := \operatorname{Ran}(f)$) is continuous at $f(x)$. Assuming $f'(x) \neq 0$, prove that f^{-1} is also differentiable at $f(x)$ and

$$(f^{-1})'\big(f(x)\big) = \frac{1}{f'(x)}$$

Hint: Consider writing $f(z) - f(x) = [f'(x) + u_x(z)](z - x)$.

Problem 2: Prove the following comparison of solutions to ordinary differential equations (ODE) known also as a way to solve differential inequalities: Let $F, G\colon \mathbb{R} \to \mathbb{R}$ (with $\operatorname{Dom}(F) = \operatorname{Dom}(G) = \mathbb{R}$) be continuous functions with

$$\forall u, v \in \mathbb{R}\colon\ u \leqslant v \ \Rightarrow\ F(u) \leqslant G(v).$$

Let $y, z\colon [a, b] \to \mathbb{R}$ be functions (defined and) continuous on $[a, b]$ and differentiable on (a, b) that solve the ODEs

$$\forall x \in (a, b)\colon\ y'(x) = F\big(y(x)\big)\ \wedge\ z'(x) = G\big(z(x)\big)$$

with the "initial" values such that $y(a) < z(a)$. Then

$$\forall x \in [a, b]\colon\ y(x) \leqslant z(x)$$

(If the ODEs have local uniqueness, then it suffices to assume $y(a) \leqslant z(a)$.)

Problem 3: (RUDIN) EX 15, PAGE 115 (Bounding $\sup|f'|$ via $\sup|f|$ and $\sup|f''|$)

Problem 4: (RUDIN) EX 16, PAGE 116 (Limit of f' under conditions on f and f''.)

Problem 5: (RUDIN) EX 17, PAGE 116 (Bound on f''' from 4 values of f or f'.)

Problem 6: Prove the following: Let $I \subseteq \mathbb{R}$ be an open interval, $n \in \mathbb{N}$ obey $n \geqslant 1$ and, given $a \in I$, assume that $f\colon I \to \mathbb{R}$ (with $\operatorname{Dom}(f) = I$) is $(n-1)$-times differentiable on I and n-times differentiable at a. Then

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x - a)^n} = 0$$

where $P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$ is the n-th order Taylor polynomial. (This is Taylor's Theorem in asymptotic form.)

Problem 7: (RUDIN) EX 22, PAGE 117 (Fixed points via iterations.)

Problem 8: (RUDIN) EX 25, PAGE 118 (Newton's method.)

Problem 9: (RUDIN) EX 26-27, PAGE 119 (Uniqueness of ODEs via Lipschitz cont.)

Problem 10: Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that the absolute value of the derivative $|f'|$ is Riemann integrable on $[a,b]$. (The values of f' at a and b are immaterial for this so think of them fixed to zero, for instance.) Prove that then f is of bounded variation and

$$V\big(f,[a,b]\big) = \int_a^b \big|f'(x)\big| \mathrm{d}x$$
