HW#3: due Mon 4/24/2023

This exercise practices functions of bounded variation and then the derivative. As in class, the emphasis is on the proofs rather than explicit examples as those should have receive plenty of attention and practice in calculus.

Problem 1: Given $f\colon [a,b] \to \mathbb{R}$ recall the definitions of positive variation $P\big(f,[a,b]\big)$ and negative variation $N\big(f,[a,b]\big)$ by

$$P\big(f,[a,b]\big) := \sup_{\Pi} \sum_{i=1}^{n} \big(f(t_i) - f(t_{i-1})\big)_+ \quad \text{and} \quad N\big(f,[a,b]\big) := \sup_{\Pi} \sum_{i=1}^{n} \big(f(t_i) - f(t_{i-1})\big)_-$$

where the supremum is over all partitions $\Pi = \{t_i\}_{i=1}^n$ of $[a,b]$ (with $n \in \mathbb{N}$ arbitrary). Prove that f being of bounded variation (i.e., $V(f,[a,b]) < \infty$) implies

$$P\big(f,[a,b]\big) < \infty \;\wedge\; N\big(f,[a,b]\big) < \infty$$

and

$$P\big(f,[a,b]\big) + N\big(f,[a,b]\big) = V\big(f,[a,b]\big)$$

Prove also that $t \mapsto P(f,[a,t])$ and $t \mapsto N(f,[a,t])$ are non-decreasing on $[a,b]$.

Problem 2: Let $f\colon [a,b] \to \mathbb{R}$ be written as $f = h - g$ with $h,g\colon [a,b] \to \mathbb{R}$ non-decreasing. Prove that $V(f,[a,b]) < \infty$. Then define $h_0\colon [a,b] \to \mathbb{R}$ by $h_0(t) := P(f,[a,t])$ and prove that $h - h_0$ is non-decreasing.

Problem 3: Given a function $f\colon \mathbb{R} \to \mathbb{R}$ and $a \in \operatorname{int}(\operatorname{Dom}(f))$, prove:

$$f'(a) \text{ exists } \Leftrightarrow \exists \alpha \in \mathbb{R}\colon \lim_{\delta \to 0^+} \frac{1}{\delta} \sup_{x \in (a-\delta, a+\delta)} \big|f(x) - f(a) - \alpha(x-a)\big| = 0$$

Problem 4: (Rudin) Ex 1, page 114 (No non-constant 2-Hölder functions.)

Problem 5: (Rudin) Ex 5, page 114 (Increments vanish if derivative vanishes)

Problem 6: (Rudin) Ex 6, page 114 (Conditions for $x \mapsto f(x)/x$ to be increasing.)

Problem 7: (Rudin) Ex 8, page 114 (Continuous $f' \Rightarrow$ uniform differentiability.)

Problem 8: (Rudin) Ex 11, page 115 (Direct computation of f''.)

Problem 9: (Convex functions are one-sided differentiable) Recall that the right and left derivatives of f at x_0 are defined by the right and left limits (read just the top signs or the bottom signs):

$$\frac{\mathrm{d}f}{\mathrm{d}x^{\pm}}(x_0) = f'^{\pm}(x_0) := \lim_{x \to x_0^{\pm}} \frac{f(x) - f(x_0)}{x - x_0}$$

Let $a < b$ be reals and let $f\colon [a,b] \to \mathbb{R}$ be convex in the sense that $\forall x,y \in [a,b]\, \forall \alpha \in [0,1]\colon f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y)$. Prove that

(1) $f'^{+}(x)$ exists for all $x \in [a, b)$ and $f'^{-}(x)$ exists for all $x \in (a, b]$.

(We do allow the derivatives at a and b to take $\pm\infty$-values.) Then show the following:

(2) f is continuous on (a, b) but not necessarily on $[a, b]$,
(3) $\forall x \in (a, b)\colon f'^{-}(x) \leqslant f'^{+}(x)$,
(4) $\forall x, y \in [a, b]\colon x < y \Rightarrow f'^{+}(x) \leqslant f'^{-}(y)$,

Assuming that f is continuous on $[a, b]$, prove also that

(5) the following version of Mean-Value Theorem holds

$$\exists x \in (a, b)\colon\ f'^{-}(x) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f'^{+}(x),$$

(6) $f^{'+}$ is right-continuous on $[a, b)$. In fact,

$$\forall x \in [a, b)\colon f'^{+}(x) = \lim_{z \to x^{+}} f'^{+}(z) = \lim_{z \to x^{+}} f'^{-}(z)$$

(Similarly, $f^{'-}$ is left-continuous on $(a, b]$ but this can be done by reflection.)

Now use these to solve the next problem.

Problem 10: (RUDIN) EX 14, PAGE 115 (Convexity under positivity of f''.)