HW2: due Mon 4/17/2023, 12:59PM

The purpose of this assignment is to give some practice of continuity and limits of functions.

Problem 1: (RUDIN) PAGE 100, EX 15 (Every continuous open mapping $\mathbb{R} \to \mathbb{R}$ is monotonic.)

Problem 2: Let X and Y be metric spaces with X totally bounded. Prove that for each $f\colon X \to Y$,

$$f \text{ Cauchy continuous} \;\Rightarrow\; f \text{ uniformly continuous}$$

(Cauchy continuity thus replaces continuity in non-complete setting.)

Problem 3: (RUDIN) PAGE 100, EX 14 (Fixed point for continuous $f\colon I \to I$.)

Problem 4: (RUDIN) PAGE 100, EX 17 Note: The conditions and conclusions are restricted to $x \in E$ where f has a simple discontinuity.

Problem 5: (RUDIN) PAGE 100, EX 19 (Continuity from intermediate value property.)

Problem 6: Let X be a metric or topological space. We say that X is path connected if for every $x, y \in X$ there is a continuous function $f\colon [0,1] \to \mathrm{X}$ with $\mathrm{Dom}(f) = [0,1]$ such that $f(0) = x$ and $f(1) = y$. Prove that

$$X \text{ path connected} \;\Rightarrow\; X \text{ connected}$$

(The converse to this fails.)

Problem 7: Let (X, ϱ) be a metric space and $f\colon X \to \mathbb{R}$ a function with $\mathrm{Dom}(f) = X$. Define

$$M_f(x) := \inf_{\delta > 0} \sup_{z \in B(x,\delta)} f(z)$$

and

$$m_f(x) := \sup_{\delta > 0} \inf_{z \in B(x,\delta)} f(z)$$

Prove that

$$\forall x \in X\colon \quad f \text{ continuous at } x \;\Leftrightarrow\; m_f(x) = M_f(x)$$

Note: the functions $M_f(x)$ and $m_f(x)$ are similar to $\limsup_{z \to x} f(z)$ and $\liminf_{z \to x} f(z)$ except that they allow "looking" at the value at x as well.

Problem 8: Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that

$$\forall r > 0 \, \forall \epsilon > 0\colon \; \big\{x \in \mathbb{R}\colon |x| < r \wedge |f(x)| > \epsilon\big\} \text{ is finite}$$

Prove that

$$\forall x \in \mathbb{R}\colon\ \lim_{z\to x} f(z) = 0$$

Given a sequence $\{q_n\}_{n\in\mathbb{N}}$ enumerating $\mathbb{Q}$, verify that the function

$$f(x) := \begin{cases} \frac{1}{n+1}, & \text{if } x = q_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{else,} \end{cases}$$

obeys the above condition. This extends Exercise 18 (page 100) from Rudin.

Problem 9: Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ and, given reals $a < b$, define its total variation on $[a, b]$ by

$$V\big(f, [a, b]\big) := \sup_{\Pi} \sum_{i=1}^{n} \big|f(t_i) - f(t_{i-1})\big|$$

where Π denotes a partition $\{t_i\}_{i=0}^{n}$ of $[a, b]$ such that $a = t_0 < t_1 < \cdots < t_n = b$. For any $f\colon \mathbb{R} \to \mathbb{R}$ with $V(f, [a, b]) < \infty$, prove:

(1) total variation is additive in the sense that

$$\forall t \in (a, b)\colon\ V\big(f, [a, b]\big) = V\big(f, [a, t]\big) + V\big(f, [t, b]\big)$$

(2) if f is continuous at $x \in [a, b]$, then so is v, where $v(t) := V\big(f, [a, t]\big)$.

Problem 10: Let $f\colon [a, b] \to \mathbb{R}$ and let $G_f := \{(t, f(t))\colon t \in [a, b]\}$ be its graph. Show that the curve in $\mathbb{R}^2$ constituting G_f is rectifiable (meaning: of finite Euclidean length) if and only if $V(f, [a, b]) < \infty$.