HW#1: due Mon 4/10/2023, 11:59PM

The purpose of this assignment is to give some additional practice on topics treated in 131AH; in particular, infima/suprema, limits of sequences and infinite series, metric spaces. We then add three additional problems on continuity which is the topic we will open up 131BH with.

Problem 1: Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of a set B which we note is partially ordered by the $\subseteq$ relation. In light of this (and the fact that suprema w.r.t. $\subseteq$ are achieved by unions and infima by intersections) we define

$$\limsup_{n\to\infty} A_n := \inf_{n\geqslant 0}\sup_{m\geqslant n} A_m = \bigcap_{n\geqslant 0}\bigcup_{m\geqslant n} A_m$$

$$\liminf_{n\to\infty} A_n := \sup_{n\geqslant 0}\inf_{m\geqslant n} A_m = \bigcup_{n\geqslant 0}\bigcap_{m\geqslant n} A_m$$

We also say

$$\lim_{n\to\infty} A_n \text{ exists } := \forall x\in B\,\exists n_0\in\mathbb{N}\colon\ (\forall n\geqslant n_0\colon x\in A_n)\vee(\forall n\geqslant n_0\colon x\notin A_n)$$

Prove that

$$\lim_{n\to\infty} A_n \text{ exists } \Leftrightarrow \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$$

(This shows that the use of *limes superior* and *limes inferior* for characterization/definition of the limit works even in partially ordered spaces provided suprema and infima exist.)

Problem 2: (Fekete lemma) Let $\{a_n\}_{n\geqslant 1}$ be a sequence of non-negative reals that is *sub-additive* in the sense

$$\forall m,n\geqslant 1\colon\ a_{m+n}\leqslant a_m+a_n$$

Prove that $\lim_{n\to\infty}\frac{a_n}{n}$ exists (in $\mathbb{R}$) by showing that, in fact,

$$\lim_{n\to\infty}\frac{a_n}{n} = \inf_{n\geqslant 1}\frac{a_n}{n}$$

Problem 3: Prove that

$$\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}\quad\text{as well as}\quad\sum_{n=3}^{\infty}\frac{1}{n(\log n)(\log\log n)^p}$$

converges when $p>1$ and diverges when $p\leqslant 1$. (Do not rely on the integral criterion as we have not yet started rigorous treatment of the Riemann integral.)

Problem 4: Prove the following special case of Riemann's rearrangement theorem: For each sequence $\{a_n\}_{n\in\mathbb{N}}$ of reals such that $\sum_{n=0}^{\infty} a_n$ converges conditionally, there exists a bijection $\phi\colon\mathbb{N}\to\mathbb{N}$ such that

$$\lim_{N\to\infty}\sum_{n=0}^{N} a_{\phi(n)} = +\infty$$

with the (improper) limit taken in extended reals.

Problem 5: Let (X, ρ) be a metric space and $A \subseteq X$ a (non-empty) compact set. Then for each $x \in X$ there exists a "closest element of A to x" in the sense

$$\forall x \in X\, \exists z \in A\colon\ \rho(x,z) = \inf\{\rho(x,y)\colon y \in A\}$$

Problem 6: Let (X, ϱ) be a metric space and $f\colon X \to X$ an everywhere-defined isometry — i.e., a map with $\mathrm{Dom}(f) = X$ and $\rho(f(x), f(y)) = \rho(x,y)$ for all $x, y \in X$. Prove

$$X \text{ (sequentially) compact } \Rightarrow\ \mathrm{Ran}(f) = X$$

Hint: Consider the contrapositive.

Problem 7: A faithful map of the United States lies on the dinner table in a house in Denver, CO. Prove that there is one, and only one, point on the map that lies exactly above the physical point (in US territory) it represents. *Note:* This is a word problem that requires a suitable interpretation in metric space theory. Do not assume that the North on the map points in the actual North direction at the map location.

Problem 8: (RUDIN) PAGE 98, EX 2 (For f continuous, $f(\overline{E}) \subseteq \overline{f(E)}$)

Problem 9: (RUDIN) PAGE 98, EX 4 (Continuous image of a dense set is dense.)

Problem 10: (RUDIN) PAGE 99, EX 6 (Continuity characterized by graph of function.)