PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 1, January 1979

A NEW AND CONSTRUCTIVE PROOF OF THE BORSUK-ULAM THEOREM

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ABSTRACT. The Borsuk-Ulam Theorem [1] states that if f is a continuous function from the n-sphere to n-space ($f\colon S^n \to \mathbf{R}^n$) then the equation $f(x) = f(-x)$ has a solution. It is usually proved by contradiction using rather advanced techniques. We give a new proof which uses only elementary techniques and which finds a solution to the equation. If f is piecewise linear our proof is constructive in every sense; it is even easily implemented on a computer.

The Borsuk-Ulam Theorem has applications to fixed-point theory and corollaries include the Ham Sandwich Theorem and Invariance of Domain. The method used here is similar to Eaves [2] and Eaves and Scarf [3].

We use the following notational conventions. Let $S^n = \{x = (x_0, \ldots, x_n) \in \mathbf{R}^{n+1} |$ some $x_i = \pm 1\}$, the boundary of a cube. Note that the antipodal map $\alpha\colon S^n \to S^n$, defined by $\alpha(x) = -x$, is a PL homeomorphism. We use $s, t \in \mathbf{R}$; $p, p', z \in \mathbf{R}^n$; $x, y \in S^n \subset \mathbf{R}^{n+1}$; $v \in S^n \times I \subset \mathbf{R}^{n+2}$; and by tuples such as (p, s) or (z, s, t) we mean the obvious points of $\mathbf{R}^{n+1}$ or $\mathbf{R}^{n+2}$. A singleton set, such as $\{t\}$, will be represented without brackets, t. The origin in $\mathbf{R}$, $\mathbf{R}^n$, and $\mathbf{R}^{n+1}$ will be represented by 0. We will let $G_t(x) = G(x, t)$.

THE PIECEWISE LINEAR BORSUK-ULAM THEOREM. *Let $f\colon S^n \to \mathbf{R}^n$ be a* PL *map. Then there exists an $x \in S^n$ such that $f(x) = f(-x)$.*

PROOF. Since f is PL, it is linear on each simplex of a triangulation T of S^n. Let $T \cap \alpha T$ denote the subdivision of T into convex cells obtained by intersecting each simplex of T with the image of a simplex of T under α. Then f is linear on each cell of $T \cap \alpha T$ and $T \cap \alpha T$ is invariant under α. We can subdivide $S^n \times I$ into convex cells by crossing each cell of $T \cap \alpha T$ with I.

We next subdivide these convex cells without adding new vertices to get a triangulation T^* which is still invariant under the homeomorphism

$$H = \alpha \times \mathrm{id}\colon S^n \times I \to S^n \times I$$

$(H(x, t) = (-x, t))$. To do this, order the pairs of vertices $\{v, H(v)\}$. Note

Received by the editors November 22, 1977 and, in revised form, March 16, 1978.

AMS (MOS) subject classifications (1970). Primary 55C20, 57C99, 57A15, 57A10; Secondary 90C99.

Key words and phrases. Borsuk-Ulam Theorem, Ham Sandwich Theorem, Invariance of Domain, fixed point, antipodal map, piecewise linear, simplicial methods, complementary pivot theory.

0002-9939/79/0000-0029/$01.75

that v and $H(v)$ cannot lie in a single convex cell of the subdivision. Suppose we have subdivided the $(r-1)$-skeleton. Let C be an r-cell and let v be a vertex of C from the first vertex pair meeting C. Use v to cone on the faces of C not containing v. (See [**5**, Problem 2.9] for the details of this argument.)

Choose $p \in \mathbf{R}^n$ so that $(p, 1, 1)$ lies in the interior of an n-simplex of T^*. Let $G_0(x) = f(x) - f(-x)$, for all $x \in S^n$, and let $G_1(z, s) = z - sp$ for all $(z, s) \in S^n$. Extend piecewise linearly to G: $S^n \times I \to \mathbf{R}^n$ using T^*. Note that $G(x, t) = -G(-x, t)$ and that $G_1^{-1}(0) = \{(p, 1), (-p, -1)\}$.

If $G(\sigma - S^n \times 0)$ contains 0 for any $(n-1)$-simplex σ of T^*, we make the following adjustments in G (otherwise take $p' = p$). Make no change in $G|S^n \times 0$. Adjust the values of G simultaneously on each pair $(z, s, 1)$ and $(-z, -s, 1)$ of vertices of T^*, redefining G by extending piecewise linearly using T^*, so that:

(a) $G(z, s, 1) = -G(-z, -s, 1)$, for all $(z, s, 1) \in S^n \times 1$,

(b) For some $p' \in \mathbf{R}^n$, $G_1^{-1}(0) = \{(p', 1), (-p', -1)\}$,and

(c) No $G(\sigma - S^n \times 0)$ contains 0, for any σ in T^* of dimension at most $n-1$.

(b) is achieved by making the change in G small. For (c), suppose that we are adjusting at v and $\sigma * v$ contradicts (c) while σ satisfies (c). Then any adjustment of $G(v)$ out of the plane determined by $G(\sigma * v)$ will make $\sigma * v$ satisfy (c).

Now let A be the component of $G^{-1}(0) - (S^n \times 0)$ containing $(p', 1, 1)$. A is a polygonal arc which has its other endpoint either in $S^n \times 1$ or $S^n \times 0$ (in the latter case A does not contain this endpoint). Then since $G(x, t) = -G(-x, t)$, $H(A)$ will be the component of $G^{-1}(0) - (S^n \times 0)$ containing $(-p', -1, 1)$. Either $A = H(A)$ or $A \cap H(A) = \phi$. The latter case holds since otherwise A is a closed arc and H would have to have a fixed point (by a PL version of the Intermediate Value Theorem). Hence $\mathrm{cl}(A)$ must be an arc connecting $S^n \times 1$ to $S^n \times 0$. So $\mathrm{cl}(A) \cap (S^n \times 0)$ is a solution. □

Thus the algorithm for finding a solution consists of following a polygonal arc in $G^{-1}(0)$ from $S^n \times 1$ to $S^n \times 0$. This algorithm can be implemented numerically using techniques similar to those used to implement the simplex method of linear programming. See [**2**] for details. In practice, the adjustment of G could be done in the process of following the arc. When the arc is found to intersect a simplex of dimension less than n, then G could be adjusted to remove the intersection.

COROLLARY (THE BORSUK-ULAM THEOREM). *Let f: $S^n \to \mathbf{R}^n$ be any continuous map. Then there exists an $x \in S^n$ so that $f(x) = f(-x)$.*

PROOF. Define f^k: $S^n \to \mathbf{R}^n$ by taking a triangulation of S^n of mesh less than $1/k$, setting $f^k(x) = f(x)$ at the vertices of the triangulation and extending linearly. Then $f^k \to f$ uniformly, and there exists $x_k \in S^n$ so that $f^k(x_k) = f^k(-x_k)$. It follows that a subsequence of $\{x_k\}$ converges to some x and $f(x) = f(-x)$. □

One cannot hope to generalize this result much by changing the antipodal map. For Pannwitz [4] gives an example in which $\gamma\colon S^n \to S^n$ is a homeomorphism isotopic to the antipodal map which takes antipodal points to antipodal points and there is no solution to $f(x) = f(\gamma(x))$. In fact, by changing $-\beta|\beta|$ to $-\beta|\beta|^{\varepsilon}$, $\varepsilon > 0$ at the bottom of p. 184 of [4], γ still has the above properties and can also be made arbitrary close to the antipodal map.

ADDED IN PROOF. J. C. Alexander and J. A. Yorke have independently found a constructive proof of the Borsuk-Ulam Theorem. Their result is contained in the paper *The homotopy continuation method: numerically implementable topological procedures*, Trans. Amer. Math. Soc. (to appear).

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