9. Supremum and infimum

The algebraic deficiencies described above seem to be related to the fact that the rational axis contains "holes." Filling some of these "holes" with radicals helps somewhat but (as we explained towards the end of the last section) "holes" remain even if all roots of all polynomial equations with integer coefficients are added in. As it turns out, the presence of the "holes" is closely related to another deficiency of the rationals, this time of the total ordering relation $\leqslant$. We will start discussing the relevant concepts at the general level and then specialize to rationals.

Consider a set E with an ordering $\leqslant$; i.e., a reflexive, antisymmetric and transitive relation. We do not assume that every pair of elements from E is ordered; so $\leqslant$ can be just a partial order. We then put out the following concepts:

Definition 9.1 (Upper/lower bound) *Given a set $A \subseteq E$, an element $x \in E$ is*

- *an* upper bound on A *if* $\forall y \in A\colon y \leqslant x$,
- *a* lower bound on A *if* $\forall y \in A\colon x \leqslant y$.

If A admits an upper bound, we say that it is bounded above *while if it admits a lower bound, we say that it is* bounded below. *If it admits both, then we say that A is* bounded.

Notice that being and upper (or lower) bound entails two things: First, every element of A compares to x and, second, the comparison is as stated. Here are some examples:

- Given a set F, let E be the powerset $\mathcal{P}(F)$ ordered by the set inclusion,
$$\forall A, B \in \mathcal{P}(F)\colon\ A \leqslant B \ :=\ A \subseteq B. \tag{9.1}$$
For any set $A \subseteq \mathcal{P}(F)$ of subsets of F, including the case when A is empty, the element $x := F$ is an upper bound on A and $x := \emptyset$ is a lower bound on A. (Hence, $x := F$ is the *maximal element* of $\mathcal{P}(F)$ and $x := \emptyset$ is the *minimal element*.)
- Consider the set of pairs $E := \mathbb{Q} \times \mathbb{Q}$ and define the *lexicographic order* on E via
$$(x,y) \leqslant (\tilde{x},\tilde{y}) \ :=\ x < \tilde{x} \vee \big(x = \tilde{x} \wedge y \leqslant \tilde{y}\big) \tag{9.2}$$
The set
$$A := \{(x,y) \in \mathbb{Q} \times \mathbb{Q}\colon 0 \leqslant x, y \leqslant 1\} \tag{9.3}$$
then admits lower bounds $(-1,-1)$, $(-1,0)$ and even $(0,0)$ and upper bounds $(2,2)$, $(1,2)$ and even $(1,1)$. On the other hand, the set
$$A := \{(x,y) \in \mathbb{Q} \times \mathbb{Q}\colon x + y = 0\} \tag{9.4}$$
admits no upper bound and no lower bound in E.

If a set A admits an upper bound, a natural next question is whether one can find the most efficient upper bound. We take this to mean the *least* upper bound which is the one that compares to and is less than all the other upper bounds. This, and the corresponding concept for the lower bound, is the content of:

Definition 9.2 (Supremum and infimum) *Given a set $A \subseteq E$, an element $x \in E$ is*

- *the* supremum of A *if x is the least upper bound of A, i.e.,*
$$\big(\forall y \in A\colon y \leqslant x\big) \wedge \Big(\forall z \in E\colon \big(\forall y \in A\colon y \leqslant z\big) \Rightarrow x \leqslant z\Big) \tag{9.5}$$

- *the* infimum of A *if it is the greatest lower bound of A, i.e.,*

$$\big(\forall y\in A\colon x\leqslant y\big)\wedge\Big(\forall z\in E\colon \big(\forall y\in A\colon z\leqslant y\big)\Rightarrow z\leqslant x\Big) \tag{9.6}$$

We note that, just as upper/lower bounds, a supremum/infimum may not exist. Still, the use of the definite article in the definition is justified by:

Lemma 9.3 *For every set A, there is at most one supremum and at most one infimum.*

Proof. Suppose $x\in E$ and $\tilde x\in E$ are both suprema of A. Then both x and $\tilde x$ are upper bounds on A. The fact that x is a supremum of A then forces $x\leqslant\tilde x$ while the fact that $\tilde x$ is a supremum of A forces $\tilde x\leqslant x$. The antisymmetry of $\leqslant$ then forces $\tilde x = x$. The argument for the infimum is analogous and so we omit it. □

We will henceforth write $\sup(A)$ for the supremum of A and $\inf(A)$ for the infimum of A, whenever these elements exist. To give an example where the existence can be guaranteed, we note:

Lemma 9.4 *For the setting of $E := \mathcal{P}(F)$ (with F a non-empty set) ordered by set inclusion $\subseteq$,*

$$\forall A\subseteq\mathcal{P}(F)\colon\ A\neq\varnothing\ \Rightarrow\ \Big(\sup(A)=\bigcup A\ \wedge\ \inf(A)=\bigcap A\Big) \tag{9.7}$$

In addition, we have $\sup(\varnothing)=\varnothing$ *and* $\inf(\varnothing)=F$.

Proof. Left to homework. □

The trivial inclusion $\bigcap A\subseteq\bigcup A$ implies that $\inf(A)$ is less or equal than $\sup(A)$ for A nonempty. This in fact holds very generally:

Lemma 9.5 *Let E be a set with ordering relation $\leqslant$. Then for any $A\subseteq E$ admitting both* $\sup(A)$ *and* $\inf(A)$*,*

$$A\neq\varnothing\ \Rightarrow\ \inf(A)\leqslant\sup(A) \tag{9.8}$$

Proof. Since $A\neq\varnothing$, there exists $a\in A$. But then $a\leqslant\sup(A)$ by the fact that the supremum is an upper bound and $\inf(A)\leqslant a$ by the fact that infimum is a lower bound. The transitivity of $\leqslant$ then gives the claim. □

The above shows that, for non-empty set, the infimum and supremum are ordered intuitively. However, as demonstrated by the case of $A=\varnothing$ in Lemma 9.4 this fails for $A=\varnothing$. (We will see another example of this when we discuss extended reals.)

Notice also that the notions of supremum and infimum appear quite symmetric. This is particularly true for the context of ordered fields. In general, we can still get the following:

Lemma 9.6 *Let E be a set with an ordering relation $\leqslant$. Then for all non-empty sets $A,B\subseteq E$ such that* $\inf(B)$ *and* $\sup(A)$ *exist,*

$$\Big(\forall a\in A\,\forall b\in B\colon a\leqslant b\Big)\ \Leftrightarrow\ \sup(A)\leqslant\inf(B) \tag{9.9}$$

Proof. The implication $\Leftarrow$ is obtained from $a\leqslant\sup(A)$ for each $a\in A$ and $\inf(B)\leqslant b$ for each $b\in B$, so we will focus on $\Rightarrow$ in (9.9). The premise says that every $a\in A$ is a lower bound on B. Since B admits an infimum, i.e., the greatest lower bound, we thus get

$$\forall a \in A\colon\ a \leqslant \inf(B) \tag{9.10}$$

But this means that $\inf(B)$ is an upper bound on A. Since A admits a supremum, i.e., the least upper bound, we must thus have $\sup(A) \leqslant \inf(B)$ as claimed. □

The following example of a situation where the infimum exists has been a subject of recent homework assignment:

Lemma 9.7 *Consider the naturals $\mathbb{N}$ ordered by the relation $\leqslant$. Then*

$$\forall A \subseteq \mathbb{N}\colon \quad A \neq \varnothing \;\Rightarrow\; \Big(\inf(A)\,\text{exists} \,\wedge\, \inf(A) \in A\Big) \tag{9.11}$$

There is a slick "existential" proof based on an argument by contradiction. While this would prove the above claim, we will follow a different argument which ultimately constructs the map $A \mapsto \inf(A)$ for all non-empty $A \subseteq \mathbb{N}$. (This is done in the proof of Lemma 9.8.) Having this map at our disposal will allow us to avoid reference to Axiom of Choice whenever "picking" an natural from a set thereof is needed.

Proof. First we invoke the recursion principle to construct a collection $\{X_n\colon n \in \mathbb{N}\}$ of elements of $\mathbb{N}$ such that

$$X_0 = 0 \,\wedge\, \left(\forall n \in \mathbb{N}\colon\ X_{n+1} = \begin{cases} X_n + 1, & \text{if } X_n \notin A, \\ X_n, & \text{if } X_n \in A. \end{cases}\right) \tag{9.12}$$

The underlying idea is very intuitive: We list progressively lower bounds on A starting from 0 until a first element of A is hit at which point the sequence freezes to the current value. We now make a couple of observations.

Step 1: $\forall n \in \mathbb{N}\colon X_n \leqslant n \,\wedge\, X_{n+1} \leqslant X_n + 1$.

The first part of the statement, $\forall n \in \mathbb{N}\colon X_n \leqslant n$ is proved by induction. (We omit the details.) The second part follows directly from (9.12).

Step 2: $\forall n \in \mathbb{N}\colon X_n$ is a lower bound on A.

We prove this by induction. Let $P_n := (\forall x \in A\colon X_n \leqslant x)$. Then P_0 is TRUE because $\forall x \in \mathbb{N}\colon 0 \leqslant x$. Next assume P_n is TRUE. If $X_n \notin A$ and there exists $k \in A$ with $k \leqslant X_n$, then $k < X_n$ and X_n is not a lower bound. It follows that $P_n \wedge X_n \notin A$ implies that $X_n + 1$ is a lower bound on A. But $X_{n+1} \leqslant X_n + 1$ by Step 1 and so X_{n+1} is then a lower bound on A as well. Summarizing these steps, we have shown

$$P_n \,\wedge\, X_n \notin A \;\Rightarrow\; X_{n+1} \text{ is a lower bound on } A \tag{9.13}$$

If, on the other hand, $X_n \in A$, then (9.12) gives $X_{n+1} = X_n$ and so we obtain

$$P_n \,\wedge\, X_n \in A \;\Rightarrow\; X_{n+1} \text{ is a lower bound on } A \tag{9.14}$$

Combining (9.13–9.14) proves $P_n \Rightarrow P_{n+1}$ and the claim in Step 2 holds by induction.

Step 3: $(\forall m \in A\colon X_m < m) \Rightarrow (\forall m \in \mathbb{N}\colon X_m = m)$

Suppose $\forall m \in A\colon X_m < m$ is TRUE and let $P_n := (X_n = n)$. Then P_0 is TRUE by (9.12). If now P_n is TRUE, then $X_n = n$ forces $X_n \notin A$ and then $X_{n+1} = X_n + 1 = n + 1$ by (9.12) again. Thus $P_n \Rightarrow P_{n+1}$ is TRUE and the claim follows by induction.

Now observe that the premise $\forall m \in A\colon X_m < m$ of Step 3 must be FALSE, for if it were TRUE then its consequence $\forall m \in \mathbb{N}\colon X_m = m$ along with $A \neq \emptyset$ forces its negation, and thus a contradiction. Hence we must have $\exists n \in A\colon X_n = n$. But this n is then a lower bound on A by Step 2 and it is also the infimum because any other lower bound m on A definitely obeys $m \leqslant n$. □

Lemma 9.8 *There exists a function* $f\colon \mathcal{P}(\mathbb{N}) \smallsetminus \{\emptyset\} \to \mathbb{N}$ *such that*

$$\operatorname{Dom}(f) = \mathcal{P}(\mathbb{N}) \smallsetminus \emptyset \ \wedge\ \Big(\forall A \in \mathcal{P}(\mathbb{N}) \smallsetminus \{\emptyset\}\colon\ f(A) = \inf(A)\Big) \tag{9.15}$$

In particular, the Axiom of Choice holds for subsets of $\mathbb{N}$.

Proof. We continue along with the objects and notation used in the previous proof. First we use induction to prove

$$\forall m, n \in \mathbb{N}\colon m \leqslant n \ \wedge\ X_m \in A \ \Rightarrow\ X_m = X_n \tag{9.16}$$

Indeed, fix $m \in \mathbb{N}$ and let $P_n := m \leqslant n \ \wedge\ X_m \in A \ \Rightarrow\ X_m = X_n$. The premise for P_0 is TRUE only if $m = 0 = n$ and then $X_m = X_n$ holds trivially. So P_0 is TRUE. Next assume $P_n \wedge m \leqslant n+1 \wedge X_m \in A$ to be TRUE. Then either $m = n+1$, which implies $X_m = X_{n+1}$ trivially, or $m \leqslant n$ and then $P_n \wedge X_m \in A$ implies $X_m = X_n$. But this also gives $X_n \in A$ and (9.12) forces $X_{n+1} = X_n$ proving $X_{n+1} = X_m$. This shows $P_n \Rightarrow P_{n+1}$ and so (9.16) follows by induction.

For each $A \in \mathcal{P}(\mathbb{N})$, define

$$S_A := \{m \in A\colon X_m = m\} \tag{9.17}$$

(Technically, $S_{A'}$ is the value of a function $A \mapsto S_A$ at A'.) Lemma 9.7 shows $S_A \neq \emptyset$ whenever $A \neq \emptyset$. Next we observe that this set is actually a singleton — i.e., a one-point set. Indeed, if $m, n \in S_A$, then $m = X_m \in A$ and $n = X_n \in A$ hold and (9.16) gives $m = X_m = X_n = n$.

In order to define the desired function f out of this, let

$$R := \Big\{(A, m) \in (\mathcal{P}(\mathbb{N}) \smallsetminus \{\emptyset\}) \times \mathbb{N}\colon m \in S_A\Big\} \tag{9.18}$$

This is a relation on $(\mathcal{P}(\mathbb{N}) \smallsetminus \{\emptyset\}) \times \mathbb{N}$ which by fact that S_A is a singleton for each non-empty $A \subseteq \mathbb{N}$ is (the graph of) a function f. (This is what spares us from having to make a choice.) The fact that $\operatorname{Dom}(f)\mathcal{P}(\mathbb{N}) \smallsetminus \{\emptyset\}$ and $f(A) = \inf(A)$ for all non-empty $A \subseteq \mathbb{N}$ now follow from Lemma 9.7. □

In the situation when either the infimum or the supremum of a set belongs to this set, we sometimes refer to them using different names:

Definition 9.9 *If A is a set such that* $\inf(A)$ *exists and* $\inf(A) \in A$ *we call* $\inf(A)$ *the* minimum *of A, with notation* $\min(A)$. *Similarly, if A admits* $\sup(A)$ *which belongs to A, we call* $\sup(A)$ *the* maximum *of A, with notation* $\max(A)$.

The reader should interpret these properly in other contexts. For instance, the *maximum of a function* is the supremum of all function values that, in addition, is achieved at some argument. This is not to be confused with the *maximizer*, which is the argument (there could be more than one) where the function achieves its (unique) maximum.

Returning back to the example of the ordered field of rationals, we note that some bounded sets of rationals do admit supremum and infimum, e.g.,

$$\sup(\{x \in \mathbb{Q} \colon x^2 \leqslant 4\}) = 2 \quad \wedge \quad \inf(\{x \in \mathbb{Q} \colon x^2 < 4\}) = -2. \tag{9.19}$$

But there are also sets that fail this. For instance, the set $\mathbb{Q}$ admits no supremum and $\emptyset$ admits no infimum simply because the former admits no upper bound and the latter no lower bound. However, even that is not the main obstruction:

Lemma 9.10 *The set $A := \{a \in \mathbb{Q} \colon a < 0 \vee a^2 \leqslant 2\}$ admits an upper bound in $\mathbb{Q}$ yet no supremum (in $\mathbb{Q}$).*

The proof needs the following trivial observation:

Lemma 9.11 (Archimedean property of $\mathbb{Q}$) $\forall a \in \mathbb{Q} \colon\ a > 0 \Rightarrow (\exists n \in \mathbb{N} \colon an > 1)$

Proof. Let $a \in \mathbb{Q}$ obey $a > 0$. Then $a = p/q$ for some $p, q \in \mathbb{N} \smallsetminus \{0\}$. Let $n := q + 1$. Then

$$anq = p(q+1) = pq + p \geqslant q + 1 > q, \tag{9.20}$$

where the inequality used that $p \geqslant 1$ and $q \geqslant 0$. Multiplying both sides by q^{-1}, which is positive and thus preserves the strict inequality, we get $an > 1$ as desired. □

We are now ready to give:

Proof of Lemma 9.10. Note that $1 \in A$ and $A \subseteq \{a \in \mathbb{Q} \colon a < 2\}$ and so A is non-empty and bounded from above by 2. Suppose for the sake of contradiction, that A admits a supremum $c \in \mathbb{Q}$. Then $1 \leqslant c$ and $c \leqslant 2$. Noting that, for any natural $n \geqslant 1$ we have

$$\left(c + \frac{1}{n}\right)^2 = c^2 + \frac{2c}{n} + \frac{1}{n^2} \leqslant c^2 + \frac{5}{n} \tag{9.21}$$

the inequality $c + \frac{1}{n} > c$ along with the fact that c is the supremum of A forces $c + \frac{1}{n} \notin A$ implying $(c + \frac{1}{n})^2 > 2$ and thus $c^2 + \frac{5}{n} > 2$. This rules out that $c^2 < 2$ because that would imply $\frac{5}{n} > 2 - c^2$ for all natural $n \geqslant 1$, in contradiction with Lemma 9.11.

We thus have $c^2 \geqslant 2$ which by Lemma 8.1 forces $c^2 > 2$. But then

$$\left(c - \frac{1}{n}\right)^2 = c^2 - \frac{2c}{n} + \frac{1}{n^2} \geqslant c^2 - \frac{4}{n} \tag{9.22}$$

shows that $(c - \frac{1}{n})^2 > 2$ because the opposite inequality would give $c^2 - 2 \leqslant \frac{4}{n}$ for all natural $n \geqslant 1$ again contradicting (in light of $c^2 > 2$) Lemma 9.11. Since $c - \frac{1}{n} \geqslant 0$, we conclude that $c - \frac{1}{n}$ is an upper bound on A that is strictly smaller than c, contradicting that c is the least upper bound. Hence, A admits no infimum in $\mathbb{Q}$. □

Of course, once $\sqrt{2}$ has been added to $\mathbb{Q}$, the set A in the previous proof will admit supremum with $\sup(A) = \sqrt{2}$. The absence of the infimum/supremum in $\mathbb{Q}$ is thus reduced to the absence of a rational solution to $x^2 = 2$, or the existence of "hole" at the point $\sqrt{2}$ in the rational line.