4. THE NATURALS

We are now sufficiently acquainted with set theoretical foundations to move to the definition of natural numbers. First we observe that the intuitive definition

$$\mathbb{N} := \{0, 1, 2, \dots\} \tag{4.1}$$

is not proper as there is no clear meaning to the dots. This has been recognized by G. Peano who put forward the following axiomatic definition:

Definition 4.1 (Peano axioms) *A triplet* $(\mathbb{N}, 0, S)$ *is said to be a* system of naturals *if the following five axioms hold:*

- (P1) $\mathbb{N}$ *is a set and* $0 \in \mathbb{N}$,
- (P2) S *is a function* $S\colon \mathbb{N} \to \mathbb{N}$ *with* $\mathrm{Dom}(S) = \mathbb{N}$,
- (P3) $\forall n \in \mathbb{N}\colon S(n) \neq 0$,
- (P4) $\forall n, m \in \mathbb{N}\colon S(n) = S(m) \Rightarrow n = m$,
- (P5) $\forall A \subseteq \mathbb{N}\colon 0 \in A \wedge S(A) \subseteq A \Rightarrow A = \mathbb{N}$.

The first two axioms basically identify what the objects in $(\mathbb{N}, 0, S)$ are so the real power rests with Axioms P3-P5. The element 0 is called the *zero element* while S is called the *successor function* and elements of its range are called *successors*. Axiom P3 tells us that 0 is not a successor while P4 imposes that the successor function is injective. These two properties ensure that $\mathbb{N}$ is not too small (and, in particular, that $\mathbb{N}$ is infinite) by ruling out, e.g., the set $\{0, 1, 2, \dots, 10\}$ with the successor function acting cyclically.

The most powerful axiom of all is P5, often referred to as the *Induction principle*, which ensures that $\mathbb{N}$ is not too large and guarantees many other useful facts. One of its elementary consequences is that 0 is the only element that is not a successor:

Lemma 4.2 *For any system of naturals* $(\mathbb{N}, 0, S)$, *we have* $S(\mathbb{N}) = \mathbb{N} \smallsetminus \{0\}$.

Proof. Let $A := S(\mathbb{N}) \cup \{0\}$. Then $0 \in A$ and, using $A \subseteq \mathbb{N}$

$$S(A) \subseteq S(\mathbb{N}) \subseteq A \tag{4.2}$$

By P5, we have $A = \mathbb{N}$. As $0 \notin S(\mathbb{N})$ by P3, we have $S(\mathbb{N}) = \mathbb{N} \smallsetminus \{0\}$. □

Another consequence is the ability to use proofs by induction when one verifies a statement depending on a natural first for zero and then proves the statement for n implies the statement for $n + 1$. That this is enough is the content of:

Lemma 4.3 (Proof by induction) *Let* $(\mathbb{N}, 0, S)$ *be a system of naturals and* $\{P_n \colon n \in \mathbb{N}\}$ *be (logical) propositions indexed thereby. Suppose that*

1. (Induction basis) P_0 *is* TRUE, *and*
2. (Induction step) $\forall n \in \mathbb{N}\colon P_n \Rightarrow P_{S(n)}$.

Then $\{n \in \mathbb{N}\colon P_n\} = \mathbb{N}$, *i.e.,* P_n *is* TRUE *for all* $n \in \mathbb{N}$.

Proof. Let $A := \{n \in \mathbb{N}\colon P_n\}$. By (1) we have $0 \in A$ and by (2) we have $\forall n \in \mathbb{N}\colon n \in A \Rightarrow S(n) \in A$, i.e., $S(A) \subseteq A$. By P5, $A = \mathbb{N}$ as claimed. □

The main task of this section is to show that the naturals exist. This comes in:

Theorem 4.4 (Existence of the naturals) *There is at least one system of naturals.*

Before we delve into the proof, let us make a historical note: In late 1800s a number of "proofs" of existence were put forward which all turned out to be flawed. One reason for this is that an axiomatic set theory can be cast *without* requiring existence of infinite sets (this is so called "Finite set theory"). Such a theory could not accommodate the naturals as these are necessarily infinite (in whatever meaning of this we take). We will thus have to use Axiom of infinity somewhere in the proof.

Proof of Theorem 4.4. Axiom of Infinity guarantees the existence of a set I such that

$$\emptyset \in I \ \wedge\ \forall X \in I\colon \{X\} \in I \tag{4.3}$$

With the choices $0 := \emptyset$ and $S(X) := \{X\}$ this set would satisfy P1-P4 of Peano axioms but it is too large to obey P5 as there are many subset thereof (corresponding, in a related construction, to *limit ordinals*) that are closed under S. We will thus define the naturals as the smallest set that contains $\emptyset$ and is closed under S.

Consider a collection of all such sets

$$K := \Bigl\{J \subseteq I\colon \emptyset \in J \wedge (\forall X\colon X \in J \Rightarrow \{X\} \in J)\Bigr\} \tag{4.4}$$

which exists thanks to Powerset and Separation Axioms. We then claim that

$$\mathbb{N} := \bigcap K \tag{4.5}$$

obeys $\mathbb{N} \in K$. For this note that $X \in \mathbb{N}$ is equivalent to $\forall J \in K\colon X \in J$ which is equivalent to $\forall J \in K\colon \{X\} \in J$ which then implies $\{X\} \in \mathbb{N}$. Similarly, $\emptyset \in \mathbb{N}$ because $\forall J \in K\colon \emptyset \in J$.

Next we define

$$0 := \emptyset \ \wedge\ \forall X \in \mathbb{N}\colon\ S(X) := \{X\} \tag{4.6}$$

and proceed to check that $(\mathbb{N}, 0, S)$ is a system of naturals. First we check Peano axioms P1-P4: From $\mathbb{N} \in K$ we have that $\mathbb{N}$ is a set with $\emptyset \in \mathbb{N}$, proving P1. For the same reason, S defined above is a function $\mathbb{N} \to \mathbb{N}$ with $\mathrm{Dom}(S) = \mathbb{N}$, proving P2. Axiom of Extensionality ensures that $\{X\} = \{Y\}$ implies $X = Y$ thus showing that S is injective, proving P4. The same axiom shows that $\emptyset$ is not a set in the range of S, proving P3.

It remains to prove the Induction Principle P5. For that let $A \subseteq \mathbb{N}$ be such that $\emptyset \in A$ and $S(A) \subseteq A$. This is readily checked to imply $A \in K$ and so $\mathbb{N} \subseteq A$ by (4.5). Lemma 2.3 now gives $A = \mathbb{N}$, proving P5. □

Our next task is to prove uniqueness of the naturals (up to natural *isomorphism*). This will hinge on a result that we will find useful later:

Theorem 4.5 (Recursion principle) *Given a system of naturals* $(\mathbb{N}, 0, S)$*, a set* E *and a function* $\mathfrak{h}\colon E \to E$ *with* $\mathrm{Dom}(\mathfrak{h}) = E$ *we have:*

$$\forall a \in E\ \exists\{X_n\colon n \in \mathbb{N}\} \subseteq E\colon\ X_0 = a \wedge \Bigl(\forall n \in \mathbb{N}\colon X_{S(n)} = \mathfrak{h}(X_n)\Bigr). \tag{4.7}$$

Moreover, the collection $\{X_n\colon n \in \mathbb{N}\}$ *satisfying* (4.7) *is unique — meaning that if* $\{X'_n\colon n \in \mathbb{N}\}$ *obeys the same recursions, then* $\forall n \in \mathbb{N}\colon X_n = X'_n$*.*

The purpose of the above is to give a rigorous meaning to the informal recursive definition whose first couple of steps are written as

$$\begin{aligned} &X_0 := a \\ &X_1 := \mathfrak{h}(a) && \text{where} \quad 1 := S(0) \\ &X_2 := \mathfrak{h}(\mathfrak{h}((a)) && \text{where} \quad 2 := S(1) \\ &X_3 := \mathfrak{h}(\mathfrak{h}(\mathfrak{h}((a))) && \text{where} \quad 3 := S(2) \\ &\ \vdots \quad \ddots \qquad \ddots && \qquad\qquad \ddots \end{aligned} \tag{4.8}$$

Although this sounds very plausible, the technical problem with this "construction" are the dots. Indeed, the procedure at best defines X_n "up to" any given natural n but defining that for all n simultaneously requires infinitely many iterations which cannot be formalized along the lines above.

The actual proof will avoid these ambiguities by careful use of set theory. The idea that we will consider all possible functions $f\colon \mathbb{N} \to E$ whose domain is a (finite or infinite) string of integers such that $f(S(n)) = \mathfrak{h}(f(n))$ for all $n \in \mathrm{Dom}(f)$. Then we take the union of the graphs of these functions and prove that this is the graph of a function whose domain is all of $\mathbb{N}$.

While the idea is simple, its formal execusion is rather lengthy and may appear impenetrable for those new to the subject or untrained in logical reasoning. Readers that feel overwhelmed by what is to come may consider skipping to the statement of Theorem 4.7. That being said, all readers should understand the statement of Theorem 4.5 as it will be used repeatedly throughout the course.

Proof of Theorem 4.5. Recall that a function $f\colon \mathbb{N} \to E$ technically a relation, and thus a subset of $\mathbb{N} \times E$. With this in mind we set

$$\mathcal{F} := \left\{ f \subseteq \mathbb{N} \times E\colon \begin{array}{c} f \text{ is a function } \wedge\ 0 \in \mathrm{Dom}(f) \wedge f(0) = a \\ \wedge \left(\begin{array}{l} \forall n \in \mathbb{N}\colon S(n) \in \mathrm{Dom}(f) \\ \qquad \Rightarrow \big(n \in \mathrm{Dom}(f) \wedge f(S(n)) = \mathfrak{h}(f(n))\big) \end{array} \right) \end{array} \right\}. \tag{4.9}$$

In words, $\mathcal{F}$ is the set of relations that are functions from $\mathbb{N}$ to E whose domain contains 0, contains the predecessor of all non-zero elements in its domain, take value a at 0 and assign value $\mathfrak{h}(f(n))$ to the successor of n. We now note:

Step 1: $\{(0, a)\} \in \mathcal{F}$ and so $\mathcal{F} \neq \varnothing$

Proof. Let $f = \{(0, a)\}$. Then f is (the graph of) a function with

$$\mathrm{Dom}(f) := \{0\} \quad \text{and} \quad f(0) := a. \tag{4.10}$$

It follows that $f \in \mathcal{F}$ and so $\mathcal{F} \neq \varnothing$. □

Step 2: $\forall f, g \in \mathcal{F}\ \forall n \in \mathbb{N}\colon n \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \Rightarrow f(n) = g(n)$

Proof. Pick $f, g \in \mathcal{F}$ and let

$$A := \Big\{ n \in \mathbb{N}\colon n \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \Rightarrow f(n) = g(n) \Big\} \tag{4.11}$$

Every function in $\mathcal{F}$ is defined at 0 and takes value a there so $0 \in A$. Now let $n \in A$ and consider $S(n)$. Using that an implication is vacuously TRUE if its premise is FALSE,

$$S(n) \notin \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \Rightarrow S(n) \in A. \tag{4.12}$$

is TRUE trivially. On the other hand, by the second line in the definition of $\mathcal{F}$, the assumption that $S(n) \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ implies $n \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ and

$$f\big(S(n)\big) = \mathfrak{h}\big(f(n)\big) \wedge g\big(S(n)\big) = \mathfrak{h}\big(g(n)\big) \tag{4.13}$$

But from $n \in A$ we know that $f(n) = g(n)$ and so

$$S(n) \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \;\Rightarrow\; f\big(S(n)\big) = g\big(S(n)\big). \tag{4.14}$$

It follows that $n \in A$ implies $S(n) \in A$ meaning that $S(A) \subseteq A$. By P5, we get $A = \mathbb{N}$ which is equivalent to the claim. □

Step 3: Define $\widehat{f} := \bigcup \mathcal{F}$. Then $\widehat{f} \in \mathcal{F}$.

Proof. The definition gives $\widehat{f} \subseteq \mathbb{N} \times E$. We first show that $\widehat{f}$ is (the graph of) a function. For that let $n \in \mathbb{N}$ and assume that, for some $x, y \in E$ we have $(n, x) \in \widehat{f}$ and $(n, y) \in \widehat{f}$. Then there exist $f, g \in \mathcal{F}$ such that $n \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ and $x = f(n)$ and $y = g(n)$. But step 2 then gives $f(n) = g(n)$ and so $x = y$. So $\widehat{f}$ is a function and, moreover,

$$\forall n \in \mathrm{Dom}(\widehat{f})\, \exists f \in \mathcal{F}\colon\; n \in \mathrm{Dom}(f) \;\wedge\; \widehat{f}(n) = f(n). \tag{4.15}$$

which will come handy in what follows.

We now have to check that $\widehat{f}$ lies in $\mathcal{F}$. Step 1 gives $0 \in \mathrm{Dom}(\widehat{f})$ and $\widehat{f}(0) = a$. Suppose now $n \in \mathbb{N}$ is such that $S(n) \in \mathrm{Dom}(\widehat{f})$. By (4.15) there is $f \in \mathcal{F}$ such that $S(n) \in \mathrm{Dom}(f)$ and $\widehat{f}(S(n))) = f(S(n))$. But $f \in \mathcal{F}$ and $S(n) \in \mathrm{Dom}(f)$ implies

$$n \in \mathrm{Dom}(f) \;\wedge\; f\big(S(n)\big) = \mathfrak{h}\big(f(n)\big) \tag{4.16}$$

and, since $\widehat{f}$ is the graph of a function that contains the graph of f, also

$$n \in \mathrm{Dom}(\widehat{f}) \;\wedge\; \widehat{f}(n) = f(n) \;\wedge\; \widehat{f}\big(S(n)\big) = \mathfrak{h}\big(\widehat{f}(n)\big) \tag{4.17}$$

This proves that $\widehat{f} \in \mathcal{F}$. □

Step 4: $\mathrm{Dom}(\widehat{f}) = \mathbb{N}$

Proof. Denote $A := \mathrm{Dom}(\widehat{f})$. Then $0 \in A$ by $\widehat{f} \in \mathcal{F}$. Next let $n \in A$ and assume, for the sake of contradiction, $S(n) \notin A$. By (4.15), there is $f \in \mathcal{F}$ with $n \in \mathrm{Dom}(f)$ and $S(n) \notin \mathrm{Dom}(f)$. Now consider the function $g\colon \mathbb{N} \to E$ with domain $\mathrm{Dom}(g) := \mathrm{Dom}(f) \cup \{S(n)\}$ and values given by

$$g(m) := \begin{cases} f(m), & \text{if } m \in \mathrm{Dom}(f), \\ \mathfrak{h}\big(f(n)\big), & \text{if } m = S(n). \end{cases} \tag{4.18}$$

We claim $g \in \mathcal{F}$. Clearly, g is a function with $0 \in \mathrm{Dom}(g)$ and $g(0) = a$. Next let $m \in \mathbb{N}$ be such $S(m) \in \mathrm{Dom}(g)$. Two alternatives are then possible. First, we may have $S(m) \in \mathrm{Dom}(f)$, which by $f \in \mathcal{F}$ forces $m \in \mathrm{Dom}(f)$ and

$$g\big(S(m)\big) = f\big(S(m)\big) = \mathfrak{h}\big(f(m)\big) = \mathfrak{h}\big(g(m)\big) \tag{4.19}$$

where the first equality is by $S(m) \in \mathrm{Dom}(f)$, the second by $f \in \mathcal{F}$ and the third by $m \in \mathrm{Dom}(f)$. Second, we may have $S(m) = S(n)$ which by the injectivity of S forces $m = n$ and so we get

$$g\big(S(m)\big) = g\big(S(n)\big) = \mathfrak{h}\big(f(n)\big) = \mathfrak{h}\big(g(n)\big) \tag{4.20}$$

where the first equality is by $S(m) = S(n)$, the second by definition of $g(S(n))$ and the third by $n \in \mathrm{Dom}(f)$ and the fact that $g(n) = f(n)$. But $g \in \mathcal{F}$ implies $S(n) \in \mathrm{Dom}(\widehat{f}) = A$, a contradiction. It follows that $S(A) \subseteq A$ and, by P5, $A = \mathbb{N}$. □

With the above in hand we are ready to complete the proof: The function $\widehat{f}\colon \mathbb{N} \to E$ with $\mathrm{Dom}(\widehat{f}) = \mathbb{N}$ constructed above obeys $\widehat{f}(0) = a$ and $\widehat{f}(S(n)) = \mathfrak{h}(\widehat{f}(n))$. Setting $X_n := \widehat{f}(n)$ for $n \in \mathbb{N}$ thus proves (4.7). To show uniqueness, let $\{X'_n\colon n \in \mathbb{N}\}$ be another such a family. Set $A := \{n \in \mathbb{N}\colon X_n = X'_n\}$. Then $0 \in A$ because $X_0 = a = X'_0$ and if $n \in A$, then $X_n = X'_n$ implies $X_{S(n)} = \mathfrak{h}(X_n) = \mathfrak{h}(X'_n) = X'_{S(n)}$ and so $S(n) \in A$, thus showing $S(A) \subseteq A$. Hence $A = \mathbb{N}$ by P5. □

Remark 4.6 We note that, writing E as $\mathbb{N} \times E'$ for a set E' and letting $h\colon \mathbb{N} \times E' \to \mathbb{N} \times E'$ be the function $(n, x) \mapsto (S(n), h_n(x))$ for a given collection $\{\mathfrak{h}_n\colon n \in \mathbb{N}\}$ of functions $\mathfrak{h}_n\colon E' \to E'$, Theorem 4.5 accommodates for the situation that $\{X_n\colon n \in \mathbb{N}\}$ obeys

$$X_0 = a \;\wedge\; \big(\forall n \in \mathbb{N}\colon X_{S(n)} = \mathfrak{h}_n(X_n)\big) \tag{4.21}$$

This allows that the "recursive rule" depends explicitly on the order of iteration.

We are now in a position to state and prove uniqueness of the naturals:

Theorem 4.7 (Uniqueness of the naturals) *Let $(\mathbb{N}, 0, S)$ and $(\mathbb{N}', 0', S')$ be two systems of naturals. Then there is a bijection $\phi\colon \mathbb{N} \to \mathbb{N}'$ such that*

$$\phi(0) = 0' \;\wedge\; \phi \circ S = S' \circ \phi. \tag{4.22}$$

Proof. Using Theorem 4.5 with the choices $E := \mathbb{N}'$, $a := 0'$ and $h := S'$ produces a function ϕ with domain $\mathrm{Dom}(\phi) = \mathbb{N}$ and properties (4.22). It remains to show that this (and, in fact, any such) function is a bijection.

We start by proving that ϕ is surjective. Let $A := \mathrm{Ran}(\phi)$. Then $0' \in A$ by the first part of (4.22) while the second part thereof implies

$$S'(A) = S' \circ \phi(\mathbb{N}) = \phi \circ S(\mathbb{N}) \subseteq \phi(\mathbb{N}) = A. \tag{4.23}$$

By P5 for the system $(\mathbb{N}', 0', S')$ we have $A = \mathbb{N}'$ thus showing that ϕ is onto.

Next we show that ϕ is injective. Consider the set

$$A := \Big\{n \in \mathbb{N}\colon \big(\forall m \in \mathbb{N}\colon \phi(m) = \phi(n) \Rightarrow m = n\big)\Big\} \tag{4.24}$$

The aim is to prove that $A = \mathbb{N}$. First note that if $\phi(m) = 0'$, then $m = 0$ for otherwise Lemma 4.2 gives $m = S(k)$ for some $k \in \mathbb{N}$ and

$$0' = \phi(m) = \phi \circ S(k) = S' \circ \phi(k) \in \mathrm{Ran}(S') \tag{4.25}$$

in contradiction with P3 for the system $(\mathbb{N}', 0', S')$. Since $0' = \phi(0)$ and since the above holds for all $m \in \mathbb{N}$, it follows that $0 \in A$.

Next assume that $n \in A$ and let $m \in \mathbb{N}$ be such that $\phi(S(n))) = \phi(m)$. Then the previous argument shows $m \neq 0$ and so $m \in \operatorname{Ran}(S)$, by Lemma 4.2. This means that $m = S(k)$ for some $k \in \mathbb{N}$ and $\phi(S(n)) = \phi(m)$ then rewrites into

$$S' \circ \phi(n) = S' \circ \phi(k) \tag{4.26}$$

The injectivity of S' forced by P4 for the system $(\mathbb{N}', 0', S')$ then gives $\phi(n) = \phi(k)$ which by $n \in A$ forces $n = k$ and so $S(n) = S(k) = m$. As m was arbitrary, we conclude that $S(n) \in A$ thus showing $S(A) \subseteq A$. By P5 for system $(\mathbb{N}, 0, S)$ we get $A = \mathbb{N}$ and so ϕ is indeed injective as claimed. □

Note that the above theory treats natural numbers in the abstract sense and, in particular, without reference to a specific "number system" or labeling convention. In light of our prior hands-on experience with the naturals, this may seem clumsy at first but is indispensable if we want to prove all familiar properties of standard number systems from axioms of set theory (rather than postulating them as axioms instead, as is done in many real-analysis textbooks).