23. Absolute vs conditional convergence

Here we refine the concept of convergent series into absolutely convergent and conditionally convergent series. The former notion will later be appreciated once we discuss power series in the next quarter.

23.1 Absolute convergence.

Although the convergence of infinite series reduces to the notion of convergence sequences, the fact that we are writing the relevant sequence as a sum brings up the following natural questions: Can the sum of infinitely many numbers be performed in any order? And what if some of the terms are subtracted instead of being added? Such considerations natural guide us toward the following concept:

Definition 23.1 (Absolute convergence) *We say that the infinite series* $\sum_{n=0}^{\infty} a_n$ converges absolutely *if* $\sum_{n=0}^{\infty} |a_n|$ *converges (in* $\mathbb{R}$*).*

We note that an infinite series with non-negative entries converges if and only if the sequence of partial sums is bounded. So absolute convergence is often stated as

$$\sum_{n=0}^{\infty} |a_n| < \infty. \tag{23.1}$$

The reader may also wonder why the term "convergence" is made part of the definition of "absolute convergence" as it refers to convergence of a different infinite series. That this is fine is the content of:

Lemma 23.2 *If an infinite series converges absolutely, then it converges.*

Proof. By the Cauchy criterion (Lemma 22.10), the convergence of the series $\sum_{n=0}^{\infty} |a_n|$ is equivalent to

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n, m \in \mathbb{N}\colon \quad n \geqslant m \geqslant n_0 \ \Rightarrow\ \sum_{k=m}^{n} |a_k| < \epsilon. \tag{23.2}$$

The triangle inequality for absolute value gives

$$\Bigl|\sum_{k=m}^{n} a_k\Bigr| \leqslant \sum_{k=m}^{n} |a_k|, \tag{23.3}$$

and so (23.2) implies

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n, m \in \mathbb{N}\colon \quad n \geqslant m \geqslant n_0 \ \Rightarrow\ \Bigl|\sum_{k=m}^{n} a_k\Bigr| < \epsilon. \tag{23.4}$$

By the Cauchy criterion again, $\sum_{n=0}^{\infty} a_n$ converges. □

A lot of properties of finite sums extends to infinite series as well; for instance:

Lemma 23.3 (Triangle inequality for infinite series) *Suppose $\sum_{n=0}^{\infty} a_n$ is absolutely convergent. Then*

$$\Bigl|\sum_{n=0}^{\infty} a_n\Bigr| \leqslant \sum_{n=0}^{\infty} |a_n| \tag{23.5}$$

We leave the proof of this easy lemma to homework. Note that the expression is meaningful although not very informative even without absolute convergence (we get $+\infty$ on the right-hand side); of course, we then still have to assume that $\sum_{k=0}^{\infty} a_k$ converges.

Moving forward on one of our questions above, we now note:

Theorem 23.4 *An absolutely convergent infinite series can be summed in any order with the same result. More precisely, if $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of reals such that* (23.1) *holds, then for every bijection $\phi\colon \mathbb{N}\to\mathbb{N}$,*

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{\phi(k)} = \sum_{k=0}^{\infty} a_k. \tag{23.6}$$

Proof. Let $\phi\colon \mathbb{N}\to\mathbb{N}$ be a bijection. Fix $\epsilon > 0$. Since the series $\sum_{k=0}^{n} a_n$ converges absolutely, the Cauchy criterion gives $n_0 \geqslant 1$ such that

$$\forall n,m\in\mathbb{N}\colon\quad m\geqslant n\geqslant n_0 \;\Rightarrow\; \sum_{k=n}^{m} |a_k| < \epsilon. \tag{23.7}$$

Define $m_0\in\mathbb{N}$ by

$$m_0 := \inf\bigl\{m\geqslant 0\colon \phi([0,n_0))\subseteq [0,m)\bigr\} \tag{23.8}$$

The fact that ϕ is injective then forces $n_0\leqslant m_0$ and we have $\phi([0,n_0))\subseteq[0,m_0)$. Using that ϕ is bijective we get that, for each $m\geqslant m_0$ (which implies $m\geqslant n_0$), the terms $a_0,\dots,a_{n_0-1}$ appear in both sums in

$$\sum_{k=0}^{m} a_{\phi(k)} - \sum_{k=0}^{m} a_k \tag{23.9}$$

and thus cancel out from the expression, while the terms $a_{n_0},\dots,a_{m_1}$ appear at most twice there. Using also (23.7) it follows that

$$\forall m\geqslant m_0\colon\quad \Bigl|\sum_{k=0}^{m} a_{\phi(k)} - \sum_{k=0}^{m} a_k\Bigr| \leqslant \sum_{k=n_0}^{m} 2|a_k| < 2\epsilon. \tag{23.10}$$

As $\{\sum_{k=0}^{m} a_k\}_{m\in\mathbb{N}}$ converges to $\sum_{k=0}^{\infty} a_k$, from (23.3) and (23.7) we get

$$\forall m\geqslant m_0\colon\quad \Bigl|\sum_{k=0}^{m} a_k - \sum_{k=0}^{\infty} a_k\Bigr| \leqslant \epsilon. \tag{23.11}$$

Using the triangle inequality we conclude

$$\forall m\geqslant m_0\colon\quad \Bigl|\sum_{k=0}^{m} a_{\phi(k)} - \sum_{k=0}^{\infty} a_k\Bigr| < 2\epsilon+\epsilon = 3\epsilon. \tag{23.12}$$

Since ϵ was arbitrary, this proves that $\sum_{k=0}^{\infty} a_{\phi(k)}$ converges to $\sum_{k=0}^{\infty} a_k$. □

The fact that the order of summation does not matter for absolutely convergent series underlies the proof that various standard manipulations with finite sums apply to infinite series. One of the useful manipulations concerns the product of two infinite series:

Lemma 23.5 (Merten's theorem for Cauchy product) *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of reals such that $\sum_{n=0}^\infty a_n$ is convergent and $\sum_{n=0}^\infty b_n$ is absolutely convergent. Setting*

$$\forall n\in\mathbb{N}\colon\ \ c_n := \sum_{k=0}^n a_k b_{n-k} \tag{23.13}$$

the series $\sum_{n=0}^\infty c_n$ is then convergent as well and

$$\sum_{n=0}^\infty c_n = \Bigl(\sum_{n=0}^\infty a_n\Bigr)\Bigl(\sum_{n=0}^\infty b_n\Bigr) \tag{23.14}$$

If both $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty b_n$ converge absolutely, then so does $\sum_{n=0}^\infty c_n$.

Proof. Assume that $\sum_{n=0}^\infty b_n$ converges absolutely and $\sum_{n=0}^\infty a_n$ converges. Let $n\in\mathbb{N}$. A simple rearrangement of the sums shows

$$\sum_{k=0}^n c_k = \sum_{k=0}^n b_k \sum_{j=0}^{n-k} a_j \tag{23.15}$$

Hereby we get

$$\Bigl(\sum_{k=0}^n b_k\Bigr)\Bigl(\sum_{j=0}^n a_j\Bigr) - \sum_{k=0}^n c_k = \sum_{k=1}^n b_k \sum_{j=n-k+1}^n a_j \tag{23.16}$$

Since $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty |b_n|$ converge, given $\epsilon>0$, the Cauchy criterion gives existence of $k_0\in\mathbb{N}$ such that

$$\forall m\geqslant n\geqslant k_0\colon\ \ \Bigl|\sum_{k=n}^m a_k\Bigr|<\epsilon\ \wedge\ \sum_{k=m}^n |b_k|<\epsilon \tag{23.17}$$

The convergence also implies that

$$a := \sup_{m\geqslant n\geqslant 0}\Bigl|\sum_{k=n}^m a_k\Bigr| < \infty\ \ \wedge\ \ b := \sum_{k=0}^\infty |b_k| < \infty \tag{23.18}$$

Assuming $n\geqslant 2n_0$, we then have

$$\begin{aligned}\Bigl|\Bigl(\sum_{k=0}^n b_k\Bigr)\Bigl(\sum_{j=0}^n a_j\Bigr) - \sum_{k=0}^n c_k\Bigr| &\leqslant \sum_{k=0}^n |b_k|\Bigl|\sum_{j=n-k+1}^n a_j\Bigr| \\ &\leqslant \Bigl(\sum_{k=0}^{\lfloor n/2\rfloor}|b_k|\Bigr)\Bigl|\sum_{j=n_0}^n a_j\Bigr| + \sum_{k=\lfloor n/2\rfloor}^n |b_k|\Bigl|\sum_{j=n-k+1}^n a_j\Bigr| \leqslant b\epsilon+\epsilon a = \epsilon(a+b)\end{aligned} \tag{23.19}$$

Using $\sum_{k=0}^n a_k\to A:=\sum_{k=0}^\infty a_k$ and $\sum_{k=0}^n b_k\to B:=\sum_{k=0}^\infty b_k$, this shows

$$AB-\epsilon(a+b)\leqslant \liminf_{n\to\infty}\sum_{k=0}^n c_k \leqslant \limsup_{n\to\infty}\sum_{k=0}^n c_k \leqslant AB+\epsilon(a+b) \tag{23.20}$$

Since ϵ is arbitrary positive, this shows equality of the *limes superior* and *limes inferior* and, consequently, $\sum_{k=0}^{n} c_k \to AB$. For the class clause note that $|c_n| \leqslant \sum_{k=0}^{n} |a_k||b_{n-k}|$ so the claim follows from (23.14) with a_n replaced by $|a_n|$ and b_n by $|b_n|$.

The last clause is proved by repeating the arguments with $|a_n|$ and $|b_n|$ instead of a_n and b_n (although a shorter and more direct argument is possible). □

Remark 23.6 The previous proof is considerably easier when both series converge absolutely. Indeed, (23.16) gives

$$\begin{aligned} \left|\Bigl(\sum_{k=0}^{n} b_k\Bigr)\Bigl(\sum_{j=0}^{n} a_j\Bigr) - \sum_{k=0}^{n} c_k\right| &\leqslant \sum_{k=0}^{n} |b_k| \sum_{j=n-k+1}^{n} |a_j| \\ &\leqslant \Bigl(\sum_{k=0}^{n} |b_k|\Bigr)\Bigl(\sum_{j=\lfloor n/2\rfloor}^{n} |a_j|\Bigr) + \Bigl(\sum_{k=\lfloor n/2\rfloor}^{n} |b_k|\Bigr)\Bigl(\sum_{j=0}^{n} |a_j|\Bigr) \end{aligned} \tag{23.21}$$

and the right-hand side then tends to zero as $n \to \infty$ by the "decaying tail" property of convergence series; cf Corollary 22.12. The same argument applies for series of absolute values, which by the inequality

$$|c_n| \leqslant \sum_{k=0}^{n} |a_k||b_{n-k}| \tag{23.22}$$

also shows absolute convergence of $\sum_{k=0}^{\infty} c_k$.

23.2 Conditional convergence.

The reliance on absolute convergence in above statements is not merely a convenience of proofs. In order to demonstrate that, introduce the following concept:

Definition 23.7 (Conditional convergence) *We say that the infinite series $\sum_{n=0}^{\infty} a_n$ converges conditionally if*

$$\sum_{n=0}^{\infty} a_n \text{ converges} \quad \wedge \quad \sum_{n=0}^{\infty} |a_n| \text{ diverges} \tag{23.23}$$

We have thus separated convergent series into those that are absolutely convergent and those that are (only) conditionally convergent. An example of a conditionally convergent series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \tag{23.24}$$

Since the harmonic series diverges (see Lemma 22.8), this series definitely fails to converge absolutely. But it converges conditionally, thanks to even numbered partial sums converging in light of

$$\sum_{k=1}^{2n} \frac{(-1)^{n-1}}{k} = \sum_{k=1}^{n}\Bigl(\frac{1}{2k-1} - \frac{1}{2k}\Bigr) = \sum_{k=1}^{n} \frac{1}{(2k-1)2k} \leqslant \sum_{k=1}^{n} \frac{1}{4k^2} \tag{23.25}$$

where the series on the right converges by Lemma 22.9, and thanks to

$$\Bigl|\sum_{k=1}^{2n-1}\frac{(-1)^{n-1}}{k}-\sum_{k=1}^{2n}\frac{(-1)^{n-1}}{k}\Bigr|\leqslant\frac{1}{2n}\underset{n\to\infty}{\longrightarrow}0 \tag{23.26}$$

which shows that the odd-numbered partial sums converge to the same limit as the even-numbered ones. This example is actually a special case of a general fact:

Lemma 23.8 (Alternating series) *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence such that*

$$\forall n\in\mathbb{N}\colon 0\leqslant a_n\ \wedge\ a_{n+1}\leqslant a_n\ \wedge\ \lim_{n\to\infty}a_n=0 \tag{23.27}$$

Then the alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

We leave the easy proof of this lemma to homework while noting that although the conditions on $\{a_n\}_{n\in\mathbb{N}}$ require that $a_n\to 0$, which we know to be necessary for convergence of $\sum_{n=0}^{\infty}(-1)^n a_n$, apart from positivity and monotonicity they do not require anything else. Thus, there are many examples where $\sum_{n=0}^{\infty}a_n$ diverges while $\sum_{n=0}^{\infty}(-1)^n a_n$ converges (albeit, by definition, only conditionally).

A similar idea underlies an example which shows that we *cannot* apply the Cauchy product formula (23.14) to series neither of which converge absolutely. Indeed, taking $a_n=b_n:=(-1)^n/\sqrt{n}$ for $n=1$ and $a_0=b_0=0$ in (23.13) shows $c_0=0$ and, for $n\geqslant 1$,

$$c_n=(-1)^n\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} \tag{23.28}$$

Since at least one of k or $n-k$ is at least $\lfloor n/2\rfloor$, hereby we get

$$|c_n|\geqslant\frac{1}{\sqrt{n/2}}\sum_{k=1}^{\lfloor n/2\rfloor}k^{-1/2} \tag{23.29}$$

where (by a similar reasoning underlying the proof of Lemma 22.9) the sum is at least a constant times $\sqrt{n}$. Hence c_n does not tend to zero as $n\to\infty$ and so $\sum_{k=1}^{n}c_k$ fails to converge by Lemma 22.5.

As our last counterexample, we show that not even Theorem 23.4 holds for conditionally convergent sequences. In fact, we have:

Theorem 23.9 (Riemann's rearrangement theorem) *Suppose $\sum_{n=0}^{\infty}a_n$ converges conditionally (and thus not absolutely). Then for each $x\in\mathbb{R}$ there is a bijection $\phi\colon\mathbb{N}\to\mathbb{N}$ such that*

$$\lim_{n\to\infty}\sum_{k=0}^{n}a_{\phi(k)}=x. \tag{23.30}$$

In short, conditionally convergent series can be rearranged to converge to any real number.

The proof hinges on the following observation:

Lemma 23.10 *Suppose $\sum_{n=0}^{\infty}a_n$ converges conditionally (and thus not absolutely). Define*

$$a_n^+:=\max\{a_n,0\}\quad\text{and}\quad a_n^-:=\max\{-a_n,0\}. \tag{23.31}$$

Then

$$\sum_{n=0}^{\infty} a_n^+ \text{ diverges} \quad \wedge \quad \sum_{n=0}^{\infty} a_n^- \text{ diverges.} \tag{23.32}$$

Proof. Note that $a_n = a_n^+ - a_n^-$ while $|a_n| = a_n^+ + a_n^-$. The lack of absolute convergence means that at least one of the series in (23.32) diverges. Since

$$\sum_{n=0}^{\infty} a_n^+ = \sum_{n=0}^{\infty} a_n^- + \sum_{n=0}^{\infty} a_n \tag{23.33}$$

where the series on the right converges, once one of the series in (23.32) diverges, so must the other. □

Proof of Theorem 23.9. Pick $x \in \mathbb{R}$. The main idea is quite simple: We will start listing the non-negative terms of $\{a_n\}_{n\in\mathbb{N}}$ in the given order until their sum first exceeds x. Then we start listing the negative terms of $\{a_n\}_{n\in\mathbb{N}}$ (starting from the first one) until the sum of all terms so far first drops again under x. Then we start listing the positive terms again, and then the negative terms, etc until all terms have been listed. (That we never fail to reach x is the consequence of (23.32).) Since x is overshot by at most $|a_n|$, for a_n being the last term added, the fact that $a_n \to 0$ as implied by convergence of $\sum_{k=0}^{\infty} a_k$ then shows that the partial sums of thus rearranged series tend to x, as desired.

The formal construction of the bijection ϕ requires introduction of there auxiliary sequences $\{n_k\}_{k\in\mathbb{N}}$, $\{m_k\}_{k\in\mathbb{N}}$ and $\{s_k\}_{k\in\mathbb{N}}$. These are defined recursively by

$$n_0 := 0 \ \wedge \ m_0 := 0 \ \wedge \ s_0 := a_0 \ \wedge \ \phi(0) := 0 \tag{23.34}$$

and, for all $k \in \mathbb{N}$,

$$s_k \leqslant x \ \Rightarrow \ \begin{cases} n_{k+1} := \inf\{n > n_k \colon a_n \geqslant 0\} \ \wedge \ m_{k+1} := m_k \\ s_{k+1} := s_k + a_{n_k} \ \wedge \ \phi(k+1) := n_{k+1} \end{cases} \tag{23.35}$$

and

$$x < s_k \ \Rightarrow \ \begin{cases} n_{k+1} := n_k \ \wedge \ m_{k+1} := \inf\{m > m_k \colon a_m < 0\} \\ s_{k+1} := s_k + a_{m_{k+1}} \ \wedge \ \phi(k+1) := m_{k+1} \end{cases} \tag{23.36}$$

Here we note that (23.32) implies that $\{a_n\}_{n\in\mathbb{N}}$ has infinitely many non-negative terms and infinitely many negative terms, and so the infima in (23.35–23.36) are well defined.

It remains to check that ϕ is a bijection and that (23.30) holds. For injectivity note that $\phi(k) = \phi(j)$ implies that either $a_{\phi(k)}$ and $a_{\phi(j)}$ are both non-negative and so $\phi(k) = n_k \wedge \phi(j) = n_j$ by (23.35), or $a_{\phi(k)}$ and $a_{\phi(j)}$ are both positive and so $\phi(k) = m_k \wedge \phi(j) = m_j$ by (23.36). But $n_k = n_j$ with $k < j$ implies that $a_{n_j} < 0 \leqslant a_{n_k}$ by (23.35), while $m_k = m_j$ with $k < j$ implies $a_{m_j} \geqslant 0 > a_{m_k}$ by (23.36), a contradiction. We conclude that $\phi(k) = \phi(j)$ implies $k = j$ and so ϕ is injective.

To prove that ϕ is surjective, assume $\mathrm{Ran}(\phi) \neq \mathbb{N}$ and let $n := \inf \mathrm{Ran}(\phi)$. If $a_n \geqslant 0$, then the fact that $\{n_k\}_{k\in\mathbb{N}}$ is non-decreasing implies that it is bounded by n. This means that the alternative (23.36) occurs from some k on, showing that

$$x \leqslant s_k + \sum_{j=k+1}^{\infty} a_{m_j} = s_k + \sum_{j=m_k+1} (-a_j^-) \tag{23.37}$$

in contradiction with the second part of (23.32). The case $a_n < 0$ is handled similarly and so we omit it.

Finally, to show that the partial sums converge, let $\epsilon > 0$ and, noting that $a_n \to 0$ by the fact that the series $\sum_{n=0}^{\infty} a_n$ converges, let $q_0 \in \mathbb{N}$ be such that $\forall q \geqslant q_0\colon |a_q| < \epsilon$. Set $k_0 := \inf\{k \geqslant 1\colon n_k \geqslant q_0 \wedge m_k \geqslant q_0\}$. Then the construction ensures

$$\forall k \geqslant k_0\colon x - \epsilon \leqslant \sum_{j=0}^{k} a_{\phi(j)} \leqslant x + \epsilon \tag{23.38}$$

and so we get the desired claim. □

23.3 Tests for absolute convergence.

We finish this section by listing some criteria for proving absolute convergence known, very likely, already from Calculus. The first one is:

Lemma 23.11 (Comparison test) *Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences such that*

$$\Bigl(\forall n \in \mathbb{N}\colon\ |a_n| \leqslant b_n\Bigr) \quad \wedge \quad \sum_{n=0}^{\infty} b_n < \infty. \tag{23.39}$$

Then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

Proof. This follows from Lemma 22.7 with a_n replaced by $|a_n|$ (and $b_n := 0$). □

A first try at the dominating sequence is the geometric progression. This leads to two limit criteria well-known called the Ratio and Root Test in calculus (albeit generalized by invoking *limes superior* instead of a plain limit). Let us start with:

Lemma 23.12 (Ratio test, convergence part) *Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence such that*

$$\forall n \in \mathbb{N}\colon\ a_n \neq 0 \tag{23.40}$$

and

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1. \tag{23.41}$$

Then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

Proof. The properties of *limes superior* implies

$$\exists n_0 \in \mathbb{N}\colon \quad q := \sup_{n\geqslant n_0}\left|\frac{a_{n+1}}{a_n}\right| < 1. \tag{23.42}$$

By induction we then infer

$$\forall n \geqslant n_0\colon \quad |a_n| \leqslant q^{n-n_0}|a_{n_0}| = q^n\bigl(q^{-n_0}|a_{n_0}|\bigr) \tag{23.43}$$

Since $|a_n| = q^n(q^{-n}|a_n|)$, hence we get

$$\forall n \in \mathbb{N}\colon \quad |a_n| \leqslant q^n \max_{k=0,\dots,n_0}\bigl(q^{-k}|a_k|\bigr) \tag{23.44}$$

Denoting the term on the right-hand side by c_n, the fact that $q < 1$ ensures that $\sum_{n=0}^{\infty} c_n$ converges. By Lemma 23.11, the series $\sum_{n=0}^{\infty} a_n$ converges absolutely. □

The Ratio Test is inconvenient for two reasons: First, we need to require that $a_n \neq 0$. Second, the failure of the condition (23.41) does not signify divergence of the series. Indeed, defining

$$\forall n \in \mathbb{N}\colon \quad a_{2n} = \left(\frac{1}{3}\right)^{2n} \quad \wedge a_{2n+1} := \left(\frac{1}{2}\right)^{2n} \tag{23.45}$$

then the series $\sum_{n=0}^{\infty} a_n$ converges absolutely yet the *limes superior* in (23.41) is infinite. This can be mended by requiring that the limit actually exists:

Lemma 23.13 (Ratio test, divergence part) *Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence such that*

$$\forall n \in \mathbb{N}\colon \ a_n \neq 0 \tag{23.46}$$

and, assuming the following limit exists,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1. \tag{23.47}$$

Then $\sum_{n=0}^{\infty} a_n$ diverges.

We leave the proof of this lemma to the reader. Instead we move to:

Lemma 23.14 (Root test) *Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of reals. Then*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} a_n \ \text{ converges absolutely,} \tag{23.48}$$

while

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} a_n \ \text{ diverges.} \tag{23.49}$$

Proof. As for the Ratio Test, we again dominate the series $\sum_{n=0}^{\infty} a_n$ by a geometric series. Assume first $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Then

$$\exists n_0 \in \mathbb{N}\colon \quad q := \sup_{n\geqslant n_0} \sqrt[n]{|a_n|} < 1. \tag{23.50}$$

It follows that $|a_n| \leqslant q^n$ for all $n \geqslant n_0$ and thus

$$\forall n \geqslant n_0\colon \quad |a_n| \leqslant q^n \max_{k=0,\dots,n_0} \bigl(q^{-k}|a_k|\bigr). \tag{23.51}$$

As $q < 0$, Lemma 23.11, the series $\sum_{n=0}^{\infty} a_n$ converges absolutely.

Next let us assume $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$. Then

$$\exists n_0 \in \mathbb{N}\colon \quad q := \sup_{n\geqslant n_0} \sqrt[n]{|a_n|} > 1. \tag{23.52}$$

But then $\limsup_{n\to\infty} |a_n| \geqslant \limsup_{n\to\infty} q^n = \infty$ which by Lemma 22.5 is inconsistent with convergence of $\sum_{n=0}^{\infty} a_n$. □

We remark that if the ratio test applies, then so does the root test. The root test is particularly useful for power series, i.e., series of the form $\sum_{n=0}^{\infty} a_n x^n$, where x is a real or complex variable. We will discuss these next quarter after we have covered absolute convergence.

Both ratio and root tests are based on comparison to the geometric series. Neither test is exhaustive because no conclusion is made when the *limes superior* in (23.48–23.49) equals one. In this case a more elaborate comparison (usually, to a polynomially decaying series) is made or some other analytic tools have to be invoked to decide convergence or divergence.