22. INFINITE SERIES

We will now proceed discussing an interesting application of the concept of limit of real-valued sequences to infinite series.

22.1 Definition and examples.

Let us first settle on some notation. Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers, for each $n \in \mathbb{N}$ we can recursively define the symbol $\sum_{k=0}^n a_k$ by:

$$\sum_{k=0}^{0} a_k := a_0 \;\wedge\; \Bigl(\forall n \in \mathbb{N}\colon \sum_{k=0}^{n+1} a_k := a_{n+1} + \sum_{k=0}^{n} a_k\Bigr). \tag{22.1}$$

For the resulting sequence $\Bigl\{\sum_{k=0}^{n} a_k\Bigr\}_{n\in\mathbb{N}}$ of *partial sums* we then impose:

Definition 22.1 (Infinite series) *Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ of reals, the infinite series $\sum_{k=0}^{\infty} a_k$ is said to be* convergent *(or* converges*) if $\lim_{n\to\infty}\sum_{k=0}^{n} a_k$ exists in $\mathbb{R}$. We then use the symbol of infinite series to denote the limit, i.e.,*

$$\sum_{k=0}^{\infty} a_k := \lim_{n\to\infty}\sum_{k=0}^{n} a_k. \tag{22.2}$$

If the limit does not exist (in $\mathbb{R}$), we say that the infinite series is divergent *(or* diverges*). In this case, the symbol of infinite series remains formal (i.e., without a numerical value).*

As we are basing our labeling on the naturals, we will typically "start" the summations at $n = 0$. However, other initial values of the summation come up as well with the definitions adapted accodingly.

There are very few examples for which the series is computable. One of these is the *geometric series*

$$\sum_{n=0}^{\infty} q^n \tag{22.3}$$

which (at this point) is a formal expression depending on parameter q called the *quotient*. Here we get:

Lemma 22.2 (Geometric series) *For each $q \in \mathbb{R}$ with $|q| < 1$, the geometric series* (22.3) *is convergent with*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}. \tag{22.4}$$

For $q \in \mathbb{R}$ with $|q| \geqslant 1$ the series is divergent.

Proof. Denote $s_n := \sum_{k=0}^{n} q^k$. Then $s_n + q^{n+1} = s_{n+1} = 1 + qs_n$, which gives $(1-q)s_n = 1 - q^{n+1}$. When $q = 1$ this equation contains no information but otherwise we get

$$\forall q \neq 1\, \forall n \in \mathbb{N}\colon \quad \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}\,. \tag{22.5}$$

For q with $|q| < 1$, we have $q^{n+1} \to 0$ and so the infinite series converges with the limit as in (22.4). On the other hand, for q with $|q| > 1$ as well as $q = -1$, the sequence $\{q^{n+1}\}_{n\in\mathbb{N}}$ does not converge and nor does the infinite series. The same applies to $q = 1$ (which was excluded from (22.5)) where $\sum_{k=0}^n q^k$ equals $n + 1$ that diverges as $n \to \infty$ as well. □

We note that the simplicity of the criterion for convergence of the geometric series is so simple, and the limit being readily computable, puts the geometric series at the center of many estimates and computations involving infinite series.

Building on the geometric series, our second example concerns the vary familiar expression of a real number using a decimal expansions. While quite intuitive and ubiquitous, the precise meaning of this expansion cannot be explained without the notion of the limit or, more accurately, infinite series. In order to state everything precisely, recall the symbol $\lfloor x\rfloor$ for lower-integer rounding of x defined by

$$\lfloor x\rfloor := \sup\{n \in \mathbb{Z} : n \leqslant x\}. \tag{22.6}$$

We then get:

Lemma 22.3 (Expansion of the reals) *Let $L \in \mathbb{N}$ be such that $L \geqslant 2$. For each $x \in [0, 1)$, define the sequence $\{x_n\}_{n\in\mathbb{N}}$ recursively by*

$$x_0 := x \ \wedge \ \Big(\forall n \in \mathbb{N} : x_{n+1} := Lx_n - \lfloor Lx_n\rfloor\Big) \tag{22.7}$$

and set $a_n := \lfloor Lx_n\rfloor$. Then $a_n \in \{0, 1, \dots, L-1\}$ for all $n \in \mathbb{N}$ and

$$x = \sum_{n=0}^{\infty} \frac{a_n}{L^{n+1}} \tag{22.8}$$

where the series on the right is convergent.

Proof. Notice that, since $z - \lfloor z\rfloor \in [0, 1)$ for each $z \in \mathbb{R}$, we have $x_n \in [0, 1)$ for each $n \in \mathbb{N}$. Also note that $a_n \in \{0, 1, \dots, L-1\}$ for each $n \in \mathbb{N}$. We claim

$$\forall n \in \mathbb{N}: \quad x = \frac{x_{n+1}}{L^{n+1}} + \sum_{k=0}^{n} \frac{a_k}{L^{k+1}} \tag{22.9}$$

This is checked readily for $n = 0$ and then proved by induction using the fact that

$$x_n = \frac{x_{n+1}}{L} + \frac{a_n}{L}. \tag{22.10}$$

(We leave the details to the reader.) Since $x_{n+1} \in [0, 1)$, from (22.9) we get

$$x - \frac{1}{L^{n+1}} \leqslant \sum_{k=0}^{n} \frac{a_k}{L^{k+1}} \leqslant x. \tag{22.11}$$

As $L > 1$, the left-hand side converges to x. By Lemma 14.7 and the Squeeze Theorem (Lemma 21.8), so do the partial sums in the middle. □

The construction in (22.7) can be easily visualized with the help of the long-division algorithm: the a_n's are the digits extracted in progressive divisions by L and x_n's are the corresponding remainders. The number $x \in [0,1)$ can then be represented by a sequence of digits from $\{0,\dots,L-1\}$ written as

$$0.a_0a_1a_2\dots \tag{22.12}$$

All $x \in \mathbb{R}$ can written this way by adding the integer $\lfloor x \rfloor$ to the expression representing the number $x - \lfloor x \rfloor$.

We note that the construction (22.7) never outputs a sequence $\{a_n\}_{n\in\mathbb{N}}$ that ends with an infinite run of $(L-1)$'s. For instance, for base-10 expansions ($L := 10$), the number $0.099999\dots$ will thus never arise; instead, we get $0.1000\dots$ right away in the first step of the long division. (Prove this!) The map $x \mapsto \{a_n\}_{n\in\mathbb{N}}$ taking $[0,1)$ into $\{0,\dots,L-1\}^{\mathbb{N}}$ is thus not onto and $\{a_n\}_{n\in\mathbb{N}} \mapsto x$ defined by (22.8) is not injective. But the defect is not too serious as it concerns only a countable set (not even all rationals).

Another interesting aspect of decimal expansions is the subject of:

Lemma 22.4 *Let $x \in [0,1)$ and the sequence $\{a_n\}_{n\in\mathbb{N}} \in \{0,\dots,L-1\}^{\mathbb{N}}$ be as in Lemma 22.3. The following are equivalent:*

(1) *x is rational, $x \in \mathbb{Q}$,*
(2) *$\{a_n\}_{n\in\mathbb{N}}$ is eventually periodic, i.e.,*

$$\exists n_0 \in \mathbb{N}\, \exists p \in \mathbb{N}\, \forall n \in \mathbb{N}\colon\ p > 1 \ \wedge\ \big(n \geqslant n_0 \ \Rightarrow\ a_{n+p} = a_n\big) \tag{22.13}$$

Thus, the number $0.21\overline{345}$ is rational but $0.101001000100001000001\dots$ is not.

We also note that the above expansion is not the only way to represent reals by sequences of naturals. Another such representation (restricted to irrationals) is the continued-fraction expansion,

$$x = \cfrac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}} \tag{22.14}$$

which is for $x \in (0,1) \smallsetminus \mathbb{Q}$ defined by

$$x_0 := x \ \wedge\ \forall n \in \mathbb{N}\colon\ x_{n+1} := 1/x_n - \lfloor 1/x_n \rfloor \tag{22.15}$$

and setting $a_n := \lfloor 1/x_n \rfloor$. (The restriction to $x \notin \mathbb{Q}$ ensures that $x_n \neq 0$ and $a_n \geqslant 1$ for all $n \in \mathbb{N}$.) While this may appear somewhat similar to the decimal expansions, there is no connection to infinite series.

22.2 Criteria for convergence.

As noted above, most infinite series are not explicitly computable. (One standard exception is the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ which can be computed using partial-fraction expansion. Do it!) Therefore, in order to determine whether a series converges one has to resort to various general criteria. We will now discuss a few of these, starting with:

Lemma 22.5 (Necessary conditions for convergence) *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of reals such that $\sum_{n=0}^{\infty} a_n$ converges. Then*

$$\lim_{n\to\infty} a_n = 0. \tag{22.16}$$

Proof. Let $s_n := \sum_{k=0}^{n} a_k$. Then $a_n = s_n - s_{n-1}$. Under the assumption of convergence, the limit $L := \lim_{n\to\infty} s_n$ exists and equals the value of the infinite series. From the addition/subtraction rule for limits, we then get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0. \tag{22.17}$$

This is the desired claim. □

We warn the reader that this is a *necessary* condition for convergence. Such conditions are typically used to rule out convergence, rather than to prove it. A *sufficient* condition for convergence is provided in:

Lemma 22.6 (Boundedness suffices for positive coefficients) *Suppose $\{a_n\}_{n\in\mathbb{N}}$ is such that $a_n \geqslant 0$ for each $n \in \mathbb{N}$. Then*

$$\sum_{n=0}^{\infty} a_n \text{ converges} \quad\Leftrightarrow\quad \left\{\sum_{k=0}^{n} a_k\right\}_{n\in\mathbb{N}} \text{ is bounded.} \tag{22.18}$$

Proof. The positivity requirement ensures that the sequence on the right of (22.18) is non-decreasing. Non-decreasing sequences converge if and only if they are bounded. □

Somewhat more useful is:

Lemma 22.7 (Comparison test) *Suppose $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $\{c_n\}_{n\in\mathbb{N}}$ are sequences with*

$$\forall n \in \mathbb{N}\colon \quad 0 \leqslant b_n \leqslant a_n \leqslant c_n. \tag{22.19}$$

Then

$$\sum_{n=0}^{\infty} c_n \text{ converges} \quad\Rightarrow\quad \sum_{n=0}^{\infty} a_n \text{ converges} \tag{22.20}$$

and

$$\sum_{n=0}^{\infty} b_n \text{ diverges} \quad\Rightarrow\quad \sum_{n=0}^{\infty} a_n \text{ diverges} \tag{22.21}$$

Proof. From (22.19) we have

$$\forall n \in \mathbb{N}\colon \quad \sum_{k=0}^{n} b_k \leqslant \sum_{k=0}^{n} a_k \leqslant \sum_{k=0}^{n} c_k. \tag{22.22}$$

The partial sums of series with non-negative coefficients form non-decreasing sequences which converge if and only if they are bounded. This readily yields (22.20–22.21). □

Using the comparison criterion, we readily conclude that the infinite series (22.8) converges for any choice of $\{a_n\}_{n\in\mathbb{N}}$ satisfying $a_n \in \{0, 1, \dots, L-1\}$. Another use of the Comparison Test produces:

Lemma 22.8 (Harmonic series) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Proof. The idea of the proof is to bound the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \dots \tag{22.23}$$

from below by the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \dots \tag{22.24}$$

and then notice that each block with the same denominator adds up to $1/2$.

Formally, this is done as follows: First note that for each $n \in \mathbb{N}$ with $n \geqslant 1$ there is exactly one $k \in \mathbb{N}$ such that $2^k \leqslant n < 2^{k+1}$. Then

$$2^k \leqslant n < 2^{k+1} \quad \Rightarrow \quad \frac{1}{n} > \frac{1}{2^{k+1}} \tag{22.25}$$

and so for all $m \geqslant 1$,

$$\sum_{n=1}^{2^m-1} \frac{1}{n} = \sum_{k=0}^{m-1} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} \geqslant \sum_{k=0}^{m-1} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{2^{k+1}} = \sum_{k=0}^{m-1} 2^k \frac{1}{2^{k+1}} = \frac{1}{2}m \tag{22.26}$$

The right hand side diverges as $m \to \infty$, which means that the sequence of partial sums for the harmonic contains a diverging subsequence, and is thus diverging itself. $\square$

The same type of reasoning then also gives:

Lemma 22.9 *For each $p > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.*

Proof. We follow a similar reasoning as in the previous lemma, but now aiming to prove convergence. Indeed, for any $m \geqslant 1$,

$$\sum_{n=1}^{2^m-1} \frac{1}{n^p} = \sum_{k=0}^{m-1} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n^p} \leqslant \sum_{k=0}^{\infty} (2^{1-p})^k \tag{22.27}$$

The geometric series on the right converges because its quotient 2^{1-p} has absolute value less than one, due to $p > 1$. $\square$

Note that, despite the sequence of coefficients decaying only *polynomially*, in both cases we ended up comparing the series to the geometric series (where *exponential* decay/growth is of concern). As noted earlier, this is a very common approach — and, usually, the first one to try — as it is guided by the fact that the geometric series has a simple convergence criterion and/or is explicitly computable.

We also note that the situation around the harmonic series can be further refined using similar methods. Indeed, we thus show that

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \text{ diverges} \quad \text{yet} \quad \forall p > 1\colon \ \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \text{ converges} \tag{22.28}$$

The case on the left can be further refined by adding $\log \log n$ terms, etc.

The next criterion is based on the equivalence of convergence and being Cauchy:

Lemma 22.10 (Cauchy criterion) *Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of reals. Then*

$$\sum_{n=0}^{\infty} a_n \text{ converges} \quad \Leftrightarrow \quad \forall \epsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall m \geqslant n \geqslant n_0\colon \quad \Big| \sum_{k=n}^{m} a_k \Big| < \epsilon. \tag{22.29}$$

Proof. The convergence of $\sum_{n=0}^{\infty} a_n$ is defined by the existence of the limit of $\{\sum_{k=0}^{n} a_k\}_{n\in\mathbb{N}}$. This is equivalent to the sequence of partial sums being Cauchy. As

$$\sum_{k=0}^{m} a_k - \sum_{k=0}^{n-1} a_k = \sum_{k=n}^{m} a_k, \tag{22.30}$$

that is in turn equivalent to the condition on the right of (22.29). □

As a consequence of this we get:

Corollary 22.11 (Finite changes are irrelevant for convergence) *If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences such that*

$$\big\{n \in \mathbb{N} \colon a_n \neq b_n\big\} \text{ is finite} \tag{22.31}$$

then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \quad\Leftrightarrow\quad \sum_{n=0}^{\infty} b_n \text{ converges.} \tag{22.32}$$

Proof. The Cauchy criterion is not affected by changing the underlying sequence on a naturals bounded by some n' since we can always take n_0 larger than n'. □

Another consequence is:

Corollary 22.12 (Decaying tail) *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of reals. Then*

$$\sum_{n=0}^{\infty} a_n \text{ converges} \;\Rightarrow\; \Big(\forall n \in \mathbb{N} \colon \sum_{k=n}^{\infty} a_k \text{ converges}\Big) \wedge \lim_{n\to\infty} \sum_{k=n}^{\infty} a_k = 0. \tag{22.33}$$

Proof. Suppose $\sum_{n=0}^{\infty} a_n$ converges. Changing the first n terms of the underlying sequence to zero, Corollary 22.11 shows that $\sum_{k=n}^{\infty} a_k$ converges for each $n \in \mathbb{N}$. Then $|\sum_{k=n}^{m} a_k| \leqslant \epsilon$ for all $m \geqslant n$ implies $|\sum_{k=n}^{\infty} a_k| \leqslant \epsilon$. The gives the limit in (22.33). □

We will give other criteria for convergence when we discuss the notions of absolute and conditional convergence in the next lecture.