21. LIMSUP AND LIMINF

In the remaining lectures of this quarter, we will return to the subjects that are more familiar from calculus. Our first step is to finish some aspects of convergence in $\mathbb{R}$ that have been overshadowed by our treatment of metric spaces. The first step towards this goal is to introduce the notions of upper and lower limits of a sequence.

21.1 The extended reals.

A defining property of the reals ensure that every non-empty set of reals that admits an upper bound admits a supremum. The two stated conditions on the set are actually somewhat related; indeed, by definition, every $a \in \mathbb{R}$ is an upper bound on $\emptyset$ but there is no supremum of $\emptyset$ because the set of all upper bounds of $\emptyset$, which is simply $\mathbb{R}$, does not have a least element. In order to overcome this issue we give:

Definition 21.1 *The set of* extended reals $\overline{\mathbb{R}}$ *is defined as*

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\} \tag{21.1}$$

where $+\infty$ *and* $-\infty$ *are elements called* positive *and* negative infinity *that obey*

$$+\infty \notin \mathbb{R} \ \wedge\ -\infty \notin \mathbb{R} \ \wedge\ +\infty \neq -\infty \tag{21.2}$$

The ordering relation $\leqslant$ *on* $\mathbb{R}$ *is then extended to* $\overline{\mathbb{R}}$ *by*

$$-\infty \leqslant \infty \ \wedge\ \big(\forall a \in \mathbb{R}\colon\ -\infty \leqslant a \wedge a \leqslant \infty\big). \tag{21.3}$$

It is readily checked that $\leqslant$ is a total ordering of $\overline{\mathbb{R}}$ with $+\infty$ being the maximal element and $-\infty$ being the minimal element. As a consequence, every subset $A \subseteq \overline{\mathbb{R}}$ now admits at least one upper bound and at least one lower bound. The sets $A \subseteq \mathbb{R}$ that admit no upper bound in $\mathbb{R}$ then have their supremum equal to positive infinity,

$$\forall A \subseteq \mathbb{R}\colon \neg\big(\exists x \in \mathbb{R}\, \forall a \in A\colon a \leqslant x\big) \Rightarrow \sup(A) = +\infty \tag{21.4}$$

while the sets without a lower bound in $\mathbb{R}$ have their infimum equal to negative infinity,

$$\forall A \subseteq \mathbb{R}\colon \neg\big(\exists x \in \mathbb{R}\, \forall a \in A\colon x \leqslant a\big) \Rightarrow \sup(A) = -\infty \tag{21.5}$$

Note that, although (21.4–21.5) do not include the case when $A = \emptyset$, because there each $x \in \mathbb{R}$ (in fact, each $x \in \overline{\mathbb{R}}$) is an upper bound as well as a lower bound, we still get

$$\sup(\emptyset) = -\infty \ \wedge\ \inf(\emptyset) = +\infty \tag{21.6}$$

by the minimality, resp., maximality of $-\infty$, resp., $+\infty$ in $\overline{\mathbb{R}}$. Sets that do contain $+\infty$ have only $+\infty$ as an upper bound, and so the supremum equals $+\infty$ for these. Similarly for the infimum of sets that contain $-\infty$.

In summary, we have proved:

Lemma 21.2 *Every* $A \subseteq \overline{\mathbb{R}}$ *admits a supremum and an infimum in* $\overline{\mathbb{R}}$.

The introduction of the two infinities to $\mathbb{R}$ is very convenient for the ordering and it preserves most of the intuitive properties we usually associate with these concepts in the reals. For instance we have

$$\forall A, B \subseteq \overline{\mathbb{R}}\colon A \subseteq B \Rightarrow \Big(\inf(B) \leqslant \inf(A) \ \wedge\ \sup(A) \leqslant \sup(B)\Big) \tag{21.7}$$

and

$$\forall A \subseteq \overline{\mathbb{R}}\colon A \neq \emptyset \Rightarrow \inf(A) \leqslant \sup(A) \tag{21.8}$$

with the warning that the conclusion actually fails for A empty, due to (21.6).

Unfortunately, the situation is more complicated once algebraic operations with infinities are needed. Many standard operations remain defined; for instance,

$$\forall a \in \mathbb{R}\colon a + (+\infty) := +\infty \wedge a + (-\infty) := -\infty \tag{21.9}$$

and

$$\forall a \in \mathbb{R}\colon a > 0 \Rightarrow a \cdot (\pm\infty) := \pm\infty \tag{21.10}$$

and

$$\forall a \in \mathbb{R}\colon a < 0 \Rightarrow a \cdot (\pm\infty) := \mp\infty \tag{21.11}$$

where, by convention, we either read only the top signs or only the bottom signs from all $\pm$ and $\mp$ on the same line. We also define

$$(+\infty) + (+\infty) := +\infty \wedge (-\infty) + (-\infty) := -\infty \tag{21.12}$$

which show $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$, and

$$(+\infty) \cdot (+\infty) := +\infty \wedge (+\infty) \cdot (-\infty) := -\infty \tag{21.13}$$

If need arises, we might at times also stipulate that

$$(\pm\infty)^{-1} := 0 \tag{21.14}$$

but this is not in the sense of the inverse element under multiplication. However, $\overline{\mathbb{R}}$ is no longer a field because expressions

$$+\infty + (-\infty), \quad -\infty + (+\infty), \quad 0 \cdot (\pm\infty) \tag{21.15}$$

are left *undefined*. In any case, the reader is caution to perform all algebraic operations involving the two infinities with extreme caution as errors are made easily.

21.2 Upper and lower limits.

Having extended supremum and infimum to all subsets of extended reals, we will now apply these concepts to sequences. Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ of extended reals, for each $n \in \mathbb{N}$ we define the symbols

$$\sup_{m\geqslant n} a_m := \sup\{a_m\colon m \in \mathbb{N} \wedge n \leqslant m\} \tag{21.16}$$

and

$$\inf_{m\geqslant n} a_m := \inf\{a_m\colon m \in \mathbb{N} \wedge n \leqslant m\} \tag{21.17}$$

From $\{a_n\}_{n\in\mathbb{N}}$ we have thus generated the sequences of its suprema and infima,

$$\{\sup_{m\geqslant n} a_m\}_{n\in\mathbb{N}} \quad \text{and} \quad \{\inf_{m\geqslant n} a_m\}_{n\in\mathbb{N}}, \tag{21.18}$$

that are non-increasing and non-decreasing, respectively, and will thus converge provided they are bounded. This is leads to:

Definition 21.3 (Limsup and liminf) *Given a sequence* $\{a_n\}_{n\in\mathbb{N}} \in \overline{\mathbb{R}}^{\mathbb{N}}$*, we define its* limes superior*, a.k.a. upper limit or limsup, by*

$$\limsup_{n\to\infty} a_n := \inf_{n\geqslant 0} \sup_{m\geqslant n} a_m \tag{21.19}$$

and its limes inferior*, a.k.a. lower limit or liminf, by*

$$\liminf_{n\to\infty} a_n := \sup_{n\geqslant 0} \inf_{m\geqslant n} a_m \tag{21.20}$$

Both upper and lower limits generally take values in $\overline{\mathbb{R}}$ even if $\{a_n\}_{n\in\mathbb{N}}$ is $\mathbb{R}$-valued. Note also that, by the monotonicity of the sequences (21.18) $n \geqslant 0$ in (21.19–21.20) could be replaced by $n \geqslant k$ for any $k \in \mathbb{N}$, and so the quantities depend only on the asymptotic properties of $\{a_n\}_{n\in\mathbb{N}}$ (meaning that changing any finite number of elements will not affect the upper and lower limits). The quantities are also naturally ordered:

Lemma 21.4 *For any* $\{a_n\}_{n\in\mathbb{N}} \in \overline{\mathbb{R}}^{\mathbb{N}}$*,*

$$\liminf_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} a_n \tag{21.21}$$

Proof. We claim that

$$\forall n, k \in \mathbb{N}\colon\ \inf_{m\geqslant k} a_m \leqslant \sup_{m\geqslant n} a_m \tag{21.22}$$

To prove this we first note that the conclusion of (21.22) is TRUE if $n = k$ by (21.8). Now if $k \leqslant n$, we use this to get

$$\inf_{m\geqslant k} a_m \leqslant \inf_{m\geqslant n} a_m \leqslant \sup_{m\geqslant n} a_m \tag{21.23}$$

which holds by (21.7) because the set of indices involved in the first infimum is larger than that in the second infimum, while for $n \leqslant k$ we use

$$\inf_{m\geqslant k} a_m \leqslant \sup_{m\geqslant k} a_m \leqslant \sup_{m\geqslant n} a_m \tag{21.24}$$

where the same argument gives the bound between the two suprema. Since $\leqslant$ is a total ordering of $\mathbb{N}$, we have proved (21.22) in all cases.

From (21.22) we get that $\{\sup_{m\geqslant n} a_m\}_{n\in\mathbb{N}}$ are all upper bounds on $\{\inf_{m\geqslant n} a_m\}_{n\in\mathbb{N}}$. Lemma 9.6 along with (21.7) then gives

$$\begin{aligned}\sup_{n\geqslant 0} \inf_{m\geqslant n} a_m &= \sup\bigl\{\inf_{m\geqslant n} a_m \colon n \in \mathbb{N}\bigr\} \\ &\leqslant \inf\bigl\{\sup_{m\geqslant n} a_m \colon n \in \mathbb{N}\bigr\} = \inf_{n\geqslant 0} \sup_{m\geqslant n} a_m\end{aligned} \tag{21.25}$$

thus proving the desired inequality. □

21.3 Connection with convergence.

The sequences (21.18) squeeze the terms of the sequence $\{a_n\}_{n\in\mathbb{N}}$ between them, and the further we go along the sequence, the more squeezed that they get. It thus appears that equality holding in (21.21) must correspond to the sequence having a limit. This is true, albeit under the additional assumption of boundedness:

Theorem 21.5 *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of reals. Then*

$$\lim_{n\to\infty} a_n \text{ exists } \Leftrightarrow \{a_n\}_{n\in\mathbb{N}} \text{ bounded } \wedge \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n. \tag{21.26}$$

When both sides are TRUE, then the limit on the left equals the common value of limsup and liminf on the right.

Proof of $\Rightarrow$ in (21.26). Suppose that $\{a_n\}_{n\in\mathbb{N}}$ has a limit and let us call the limit L. As sequential convergence in $\mathbb{R}$ arises from a metric, Lemmas 17.4 shows that $\{a_n\}_{n\in\mathbb{N}}$ is bounded, so it suffices to prove equality of limsup and liminf. Here we note that convergence to a limit means that for all $k \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0\colon\ |a_n - L| < \frac{1}{k+1} \tag{21.27}$$

This is rewritten as

$$\forall n \geqslant n_0\colon\ L - \frac{1}{k+1} < a_n < L + \frac{1}{k+1}. \tag{21.28}$$

Using the definitions (21.19–21.20) and Lemma 21.4 it follows that for all $n \geqslant n_0$,

$$L - \frac{1}{k+1} < \inf_{m\geqslant n} a_m \leqslant \liminf_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} a_n \leqslant \sup_{m\geqslant n} a_m < L + \frac{1}{k+1}. \tag{21.29}$$

But both limsup and liminf are finite by boundedness of $\{a_n\}_{n\in\mathbb{N}}$ and so can subtract one of the other to get

$$0 \leqslant \limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n < \frac{2}{k+1} \tag{21.30}$$

By the Archimedean property of the reals (see Theorem 11.1), the only non-negative real number that is less than $\frac{2}{k+1}$ for all $k \in \mathbb{N}$ is zero and so

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n \tag{21.31}$$

as claimed on the right-hand side of (21.26). □

Proof of $\Leftarrow$ in (21.26). The argument is similar, albeit somewhat easier. Suppose $\{a_n\}_{n\in\mathbb{N}}$ is bounded and (21.31) holds. The common value L of the latter quantities is then $\mathbb{R}$-valued. Fix $k \in \mathbb{N}$. Then

$$\exists n_0 \in \mathbb{N}\colon\ \sup_{m\geqslant n_0} a_m < L + \frac{1}{k+1}, \tag{21.32}$$

for otherwise $L + \frac{1}{k+1}$ would be a better lower bound on the supremum sequence and, similarly,

$$\exists \tilde{n}_0 \in \mathbb{N}\colon\ \inf_{m\geqslant \tilde{n}_0} a_m > L - \frac{1}{k+1}. \tag{21.33}$$

But then

$$\forall m \geqslant \max\{n_0, \tilde{n}_0\}\colon\ L - \frac{1}{k+1} < a_m < L + \frac{1}{k+1} \tag{21.34}$$

Rewriting the inequalities on the right as $|a_m - L| < \frac{1}{k+1}$, we have proved that L is the limit of $\{a_n\}_{n\in\mathbb{N}}$. □

Definition 21.6 (Improper limit) *We say that a sequence $\{a_n\}_{n\in\mathbb{N}}$ of reals has an improper limit if*

$$\{a_n\}_{n\in\mathbb{N}} \text{ is unbounded} \quad \wedge \quad \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n. \tag{21.35}$$

Under (21.35), the common value of limsup and liminf is then necessarily $+\infty$ or $-\infty$ and the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded either from above or from below (but not both). This permits us to extend the notation so that:

$$\lim_{n\to\infty} a_n = +\infty \quad \text{if } \{a_n\}_{n\in\mathbb{N}} \text{ is bounded from below and (21.35) holds} \tag{21.36}$$

and

$$\lim_{n\to\infty} a_n = -\infty \quad \text{if } \{a_n\}_{n\in\mathbb{N}} \text{ is bounded from above and (21.35) holds} \tag{21.37}$$

In this case we will at times say that the limit exists in $\overline{\mathbb{R}}$.

Improper limits do not conform to the definition of the limit in $\mathbb{R}$, which would imply that the sequence is Cauchy and bounded (under Euclidean metric), both of which fail for improper limits. However, they do become proper limits when we endow $\overline{\mathbb{R}}$ with a different metric, e.g.,

$$\tilde{\varrho}(x,y) := \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| \tag{21.38}$$

where we set $\frac{\pm\infty}{1+|\pm\infty|} := \pm 1$. Indeed, in this metric $\overline{\mathbb{R}}$ is simply a completion of $\mathbb{R}$. This makes saying that the limit exists in $\overline{\mathbb{R}}$ completely consistent with our earlier definitions.

21.4 Manipulations with limits.

In order to conclude our general discussion of limits of real-valued sequences, we recall some "rules" fo computing with such limits:

Lemma 21.7 (Sum, Product and Quotient Rules) *Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are two sequences such that the limits*

$$A := \lim_{n\to\infty} a_n \ \wedge \ B := \lim_{n\to\infty} b_n \tag{21.39}$$

exist in $\overline{\mathbb{R}}$. Then:

(1) $\lim_{n\to\infty}(a_n + b_n)$ *exists and equals* $A + B$,
(2) *for any* $c \in \mathbb{R}$, $\lim_{n\to\infty} ca_n$ *exists and equals* cA,
(3) $\lim_{n\to\infty} a_n \cdot b_n$ *exists and equals* $A \cdot B$,
(4) *if* $b_n \neq 0$ *for all* $n \in \mathbb{N}$ *AND* $B \neq 0$, *then also*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B} \tag{21.40}$$

provided the expressions on the right are meaningful.

We leave the proof of these to the reader while noting that, for $A, B \in \mathbb{R}$, the requirement that the right-hand side are meaningful is trivial except in (21.40), where we need to assume $B \neq 0$. Once one or both of A and B are infinite, we have to exclude expressions of the form (21.15).

We also get the very popular tool for proving existence of a limit:

Lemma 21.8 (Squeeze Theorem) *Suppose* $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ *and* $\{c_n\}_{n\in\mathbb{N}}$ *are* $\overline{\mathbb{R}}$*-valued sequences such that*

$$\forall n \in \mathbb{N}\colon \ b_n \leqslant a_n \leqslant c_n. \tag{21.41}$$

If the limits in

$$L := \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n \tag{21.42}$$

exist in $\overline{\mathbb{R}}$. *Then*

$$\lim_{n\to\infty} a_n = L. \tag{21.43}$$

We leave proofs of these facts to a homework exercise.