20. Compactness and topology

In the previous section, we defined the notion of sequential compactness by asking that every sequences of points contain convergent subsequence. Here we will discuss the consequences of compactness for the open sets, a.k.a. topology and then explain how compactness arises in topological spaces.

20.1 Cantor's intersection property.

A classical result of Cantor says that the reals are uncountable. We showed this in Theorem 13.1 using a diagonal argument. This theorem was dated 1891, but Cantor first proved the result nearly 20 years earlier by an argument that relies, in its nature, on sequential compactness. Here is his theorem again:

Theorem 20.1 (Cantor 1874) $[0,1] := \{x \in \mathbb{R} \colon 0 \leqslant x \leqslant 1\}$ *is not countable.*

Proof. Suppose, for the sake of contradiction, that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that $[0,1] = \{x_n \colon n \in \mathbb{N}\}$. We will now construct two auxiliary sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ satisfying

$$\forall n \in \mathbb{N} \colon\ 0 \leqslant a_n < b_n \leqslant 1 \tag{20.1}$$

as follows: Set $a_0 := 0$ and $b_0 := 1$ and, assuming a_n and b_n have been defined so that (20.1) holds, define a_{n+1} and b_{n+1} by

$$\forall n \in \mathbb{N} \colon\ \begin{cases} x_n \leqslant \dfrac{a_n + b_n}{2} \ \Rightarrow\ a_{n+1} := \dfrac{a_n + 2b_n}{3} \ \wedge\ b_{n+1} := b_n \\ \dfrac{a_n + b_n}{2} < x_n \ \Rightarrow\ a_{n+1} := a_n \ \wedge\ b_{n+1} := \dfrac{2a_n + b_n}{3} \end{cases} \tag{20.2}$$

We now readily check that $\{a_n\}_{n\in\mathbb{N}}$ is non-decreasing and $\{b_n\}_{n\in\mathbb{N}}$ is non-increasing. The monotonicity upgrades (20.1) into

$$\forall n, m \in \mathbb{N} \colon n \leqslant m \Rightarrow a_n \leqslant b_m \tag{20.3}$$

and so each a_n is a lower bound on $\{b_m \colon m \in \mathbb{N}\}$ and each b_m is an upper bound on $\{a_n \colon n \in \mathbb{N}\}$. Denoting

$$a := \sup\{a_n \colon n \in \mathbb{N}\} \ \wedge\ b := \inf\{b_m \colon m \in \mathbb{N}\} \tag{20.4}$$

we thus have $0 \leqslant a \leqslant b \leqslant 1$ by an exercise in an earlier homework. (Alternatively, we can use Lemma 17.6 to show that $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ and then infer the inequality from (20.1).) But (20.1–20.2) ensure

$$\forall n \in \mathbb{N} \colon x_n \notin [a_{n+1}, b_{n+1}] \tag{20.5}$$

and, since $[a,b] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$,

$$\forall n \in \mathbb{N} \colon x_n \notin [a,b] \tag{20.6}$$

In particular, the number a, which lies in $[0,1]$, is not a member of $\{x_n\}_{n\in\mathbb{N}}$ in contradiction with the assumption that this sequence lists all points in $[0,1]$. □

The previous proof clearly uses the same idea as our proof of the Bolzano-Weierstrass theorem and could be deduced from sequential compactness of $[0,1]$. Notwithstanding, we can also recast the key argument in the previous proof as follows: Denote

$$C_n := [a_n, b_n] \tag{20.7}$$

Then the monotonicities of $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ give

$$\forall n\in\mathbb{N}\colon\ C_{n+1}\subseteq C_n \tag{20.8}$$

and so $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of nested closed non-empty subintervals of $[0,1]$. The proof then hinges on the fact that these properties imply

$$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset \tag{20.9}$$

As it turns out, this argument (suitably generalized to closed sets) applies to all sequentially compact spaces:

Theorem 20.2 (AC)(Cantor's intersection property) *A metric space (X,ϱ) is sequentially compact if and only if every nested sequence of non-empty close subsets has a non-empty intersection, i.e., for all $\{C_n\}_{n\in\mathbb{N}}\in\mathcal{P}(X)^{\mathbb{N}}$ we have*

$$\Bigl(\forall n\in\mathbb{N}\colon C_n \text{ closed} \wedge C_n\neq\emptyset \wedge C_{n+1}\subseteq C_n\Bigr) \Rightarrow \bigcap_{n\in\mathbb{N}} C_n\neq\emptyset. \tag{20.10}$$

Proof of necessity of (20.10). Assume that (X,ϱ) is sequentially compact and let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of non-empty closed sets with $C_{n+1}\subseteq C_n$ for each $n\in\mathbb{N}$. Since $C_n\neq\emptyset$, we may pick (using the Axiom of Choice) $x_n\in C_n$ for each $n\in\mathbb{N}$. The compactness of X ensures existence of convergent subsequence, $x_{n_k}\to x$. Since $n_k\geqslant k$, for each $n\in\mathbb{N}$ we have $x_{n_k}\in C_n$ as soon as $k\geqslant n$ and so, since C_n is closed and thus contains the limits of all convergent sequences, $x\in C_n$ for all $n\in\mathbb{N}$. It follows that $x\in\bigcap_{n\in\mathbb{N}} C_n$ and so the intersection is indeed non-empty. □

Proof of sufficiency of (20.10). For the converse let us now assume that, for each sequence $\{C_n\}_{n\in\mathbb{N}}\in\mathcal{P}(X)^{\mathbb{N}}$, we have (20.10). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence from X. Then

$$C_n := \overline{\{x_m\colon m\geqslant n\}} \tag{20.11}$$

are closed (by definition) and non-empty, because $x_n\in C_n$ for each $n\in\mathbb{N}$. Since $A\subseteq B$ implies $\overline{A}\subseteq\overline{B}$, we also have $C_{n+1}\subseteq C_n$ for all $n\in\mathbb{N}$. By (20.10), $\bigcap_{n\in\mathbb{N}} C_n\neq\emptyset$.

Let $x\in\bigcap_{n\in\mathbb{N}} C_n$. By the fact that the closure of the set coincides with the set of the adherent points (see Lemma 16.2), we have

$$\forall r>0\,\forall n\in\mathbb{N}\colon\ B(x,r)\cap\{x_m\colon m\geqslant n\}\neq\emptyset. \tag{20.12}$$

The sets $I_k := \{n\geqslant k\colon x_n\in B(x,2^{-k})\}$ then obey

$$\forall k\in\mathbb{N}\colon\ I_k \text{ infinite} \wedge I_{k+1}\subseteq I_k \tag{20.13}$$

Defining $\{n_k\}_{k\in\mathbb{N}}$ from $\{I_k\}_{k\in\mathbb{N}}$ as in Lemma 19.7, we get

$$\forall k\in\mathbb{N}\colon \varrho(x_{n_k},x)<2^{-k} \tag{20.14}$$

and so $x_{n_k}\to x$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ thus contains a convergent subsequence and (X,ϱ) is sequentially compact as claimed. □

20.2 Compactness via open covers.

The Cantor intersection property has the following equivalent formulation:

Theorem 20.3 (AC)(Countable open cover property) *A metric space (X, ϱ) has the Cantor intersection property* (20.10) *or, equivalently, is sequentially compact if and only if for any sequence $\{O_n\}_{n\in\mathbb{N}}$ of open subsets of X,*

$$\bigcup_{n\in\mathbb{N}} O_n = X \;\Rightarrow\; \exists n \in \mathbb{N}\colon\; \bigcup_{k=0}^{n} O_k = X \tag{20.15}$$

i.e., if and only if every countable open cover contains a finite subcover.

Proof. Note that, for any sequence $\{O_n\}_{n\in\mathbb{N}}$ of open subsets of X,

$$C_n := X \smallsetminus \bigcup_{k=0}^{n} O_k \tag{20.16}$$

defines a sequence of nested closed subsets of X. In addition, $\{O_n\}_{n\in\mathbb{N}}$ is a cover of X, i.e., $\bigcup_{n\in N} O_n = X$, if and only if $\{C_n\}_{n\in\mathbb{N}}$ have empty intersection. So (20.15) is equivalent the statement that, for any sequence $\{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}(X)^{\mathbb{N}}$:

$$\Bigl(\forall n \in \mathbb{N}\colon C_n \text{ closed} \;\wedge\; C_{n+1} \subseteq C_n\Bigr) \wedge \bigcap_{n\in\mathbb{N}} C_n = \emptyset \;\Rightarrow\; \exists n \in \mathbb{N}\colon\; C_n = \emptyset. \tag{20.17}$$

This is the contrapositive to (20.10). □

The property from the previous theorem can further be generalized as follows:

Definition 20.4 (Compactness in topology) *A topological space — i.e., a set X with a class of open sets satisfying the standard axioms — is said to be* compact *if for any set $\{O_\alpha\colon \alpha \in I\}$ of open subsets of X,*

$$\bigcup_{\alpha\in I} O_\alpha = X \;\Rightarrow\; \exists F \subseteq I\colon\; F \text{ finite} \wedge \bigcup_{\alpha\in F} O_\alpha = X \tag{20.18}$$

The difference compared to Theorem 20.3 is that here we are asking the open cover property to hold for arbitrary covers by open sets, not just countable ones. This makes a difference in general — and constitutes the distinction between *compactness* and *countable compactness* — but not for metric spaces. To explain this, recall the notion of separability from Definition 16.9. We then note:

Lemma 20.5 (AC) *Any totally bounded metric space (X, ϱ) is separable.*

Proof. The total boundedness implies

$$\forall k \in \mathbb{N}\, \exists m_k \in \mathbb{N}\, \exists z_1^{(k)}, \dots, z_{m_k}^{(k)} \in X\colon\; \bigcup_{i=0}^{m_k} B(x_i^{(k)}, 2^{-k}) = X. \tag{20.19}$$

Let

$$A := \bigcup_{k\in\mathbb{N}} \{z_1^{(k)}, \dots, z_{m_k}^{(k)}\} \tag{20.20}$$

Then, being a countable union of finite sets, A is countable by Corollary 12.13. Moreover, for each $x \in \mathbb{N}$ and each $n \in \mathbb{N}$, there is $z \in A$ — namely, $z \in \{z_1^{(n)}, \dots, z_{m_n}^{(n)}\}$ — with $\varrho(x,z) < 2^{-n}$. It follows that $\overline{A} = X$ and so X is separable. □

The fact that the distinction between general open covers and countable open covers makes no difference for metric spaces is then a consequence of:

Lemma 20.6 (Lindelöf's lemma) *Let (X,ϱ) be separable. Then any open cover of X contains a countable subcover, i.e., any class $\{O_\alpha\colon \alpha \in I\}$ of open sets,*

$$\bigcup_{\alpha\in I} O_\alpha = X \;\Rightarrow\; \exists J \subseteq I\colon\; J \text{ countable} \wedge \bigcup_{\alpha\in J} O_\alpha = X \tag{20.21}$$

Proof. Let $A \subseteq X$ be a countable dense subset. Then $A = \{x_n\colon n \in \mathbb{N}\}$ for some sequence $\{x_n\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, let

$$m(n) := \inf\bigl\{m \in \mathbb{N}\colon (\exists \alpha \in I\colon B(x_n, 2^{-m}) \subseteq O_\alpha)\bigr\}, \tag{20.22}$$

where the set under infimum is non-empty because, since $\{O_\alpha\colon \alpha \in I\}$ is a cover, the set $\{\alpha \in I\colon x \in O_\alpha\}$ is non-empty and the fact that O_α is open shows that $x_n \in O_\alpha$ implies $B(x_n, 2^{-m}) \subseteq O_\alpha$ for $m \in \mathbb{N}$ sufficiently large.

Assuming the Axiom of Choice, we now pick

$$\alpha_n \in \bigl\{\alpha \in I\colon B(x_n, 2^{-m(n)}) \subseteq O_\alpha\bigr\} \tag{20.23}$$

for each $n \in \mathbb{N}$ and claim

$$\bigcup_{n\in\mathbb{N}} O_{\alpha_n} = X. \tag{20.24}$$

Indeed, let $x \in X$. Since $\{O_\alpha\colon \alpha \in I\}$ is an open cover of X, there exist $\alpha \in I$ and $k \in \mathbb{N}$ such that $B(x, 2^{-k}) \subseteq O_\alpha$. The fact that A is dense in X in turn implies that there exists $n \in \mathbb{N}$ such that $x_n \in B(x, 2^{-k-1})$. But then

$$B(x_n, 2^{-k-1}) \subseteq B(x, 2^{-k}) \subseteq O_\alpha \tag{20.25}$$

and so $m(n) \leqslant k+1$. This in turn implies

$$x \in B(x_n, 2^{-k-1}) \subseteq B(x_n, 2^{-m(n)}) \subseteq O_{\alpha_n}. \tag{20.26}$$

Hence, every $x \in X$ satisfies $x \in \bigcup_{n\in\mathbb{N}} O_{\alpha_n}$ and so we get (20.24). □

We now have all the ingredients needed for:

Theorem 20.7 (AC) *For metric spaces, sequential compactness is equivalent to compactness.*

Proof. Since the open cover property implies the countable open cover property as a special case, the "if" part of Theorem 20.3 shows that compactness implies sequential compactness. For the converse direction, a sequentially compact metric space is separable by Lemmas 19.10 and 20.5 and so, by Lemma 20.6, any open cover can be reduced to a countable subcover. The "only if" part of Theorem 20.3 then ensures that this subcover contains a finite sub-subcover, proving compactness. □

We note that Lindelöf's lemma extends even beyond metric spaces; namely, to the spaces where the topology admits a countable base — these are called *second-countable* spaces. In general, the topological spaces for which the conclusion of Lemma 20.6 holds are called *Lindelöf spaces*. Second countability is sufficient but not necessary for being Lindelöf. The argument used in the proof of Lemma 20.6 can be used to prove the characterization of open subsets of $\mathbb{R}$; cf Theorem 15.14 which is sometimes also called Lindelöf's lemma adding prefix "generalized" to the version in Lemma 20.6.

20.3 Consequences for cardinality.

The notions of compactness (and completeness) are interestingly linked with certain cardinality considerations for metric spaces. We saw one of these in Theorem 20.1 and Lemma 20.5 but other similar connections exist. As these go beyond the scope of these lectures, we will be very brief.

We start with a definition that is already familiar from homework:

Definition 20.8 (Perfect set) *A subset A of a metric space is said to be* perfect *if it is closed and has no isolated points.*

There are many examples of perfect sets; e.g., any closed subinteral of $\mathbb{R}$ or the *Cantor ternary set*, which is the image of $\{0,1\}^{\mathbb{N}}$ under the map f from (13.5). The latter example of the Cantor ternary set is actually very typical:

Theorem 20.9 *Let (X,ϱ) be a complete metric space and $A \subseteq X$ a perfect set. Then there is an injection $f\colon \{0,1\}^{\mathbb{N}} \to A$.*

Proof (main idea). We present only the main idea. Let $x \in A$. Since A is perfect, x is not isolated and so a ball of radius 1 contains at least two points in A distinct from x, say x_0 and x_1. Letting $r_0 := \frac{1}{3}\min\{\varrho(x,x_0),\varrho(x,x_1),\varrho(x_0,x_1)\}$, the closed balls $B'(x_0,r)$ and $B'(x_1,r)$ then also contain two points each, say $x_{00}, x_{01} \in B'(x_0,r)\smallsetminus\{x_0\}$ and $x_{10}, x_{11} \in B'(x_1,r)\smallsetminus\{x_1\}$. Proceeding recursively, at level $n \in \mathbb{N}$ of the recursion, we have defined a distinct point $x_{\sigma_0\dots\sigma_n} \in A$ for each $\sigma_0,\dots,\sigma_n \in \{0,1\}$ with all these points separated by at least distance r_n. Then we set r_{n+1} to be $1/3$ of the minimum distance between all the points defined so far and then, in each closed ball $B'(x_{\sigma_0\dots\sigma_n}, r_{n+1})$, we pick two points $x_{\sigma_0\dots\sigma_n 0}$ and $x_{\sigma_0\dots\sigma_n 1}$ distinct from $x_{\sigma_0\dots\sigma_n}$.

Since $\varrho(x_{\sigma_0\dots\sigma_{n+1}}, x_{\sigma_0\dots\sigma_n}) \leqslant r_{n+1}$ and $r_n \to 0$ exponentially fast, we have

$$\forall \sigma = (\sigma_0,\sigma_1,\dots) \in \{0,1\}^{\mathbb{N}}\colon\ f(\sigma) := \lim_{n\to\infty} x_{\sigma_0\dots\sigma_n} \text{ exists} \tag{20.27}$$

with $f(\sigma) = f(\sigma')$ only if $\sigma = \sigma'$ thanks to the use of closed balls and the fact that the balls identified at level n are disjoint from one another. This is the desired injection. □

As a consequence of this we get:

Corollary 20.10 *A perfect set A has always at least the cardinality of the continuum. If the underlying metric space is separable, then A is of the cardinality of the continuum.*

Proof. Since $\{0,1\}^{\mathbb{N}}$ has the cardinality of the continuum by (13.17), any sets that embeds it injectively has at least that cardinality. On the other hand, any point x in a separable metric space is a limit of a subsequence of the dense sequence of points and so can be identified with a subset of $\mathbb{N}$. As $\mathcal{P}(\mathbb{N})$ is equinumerous to $\{0,1\}^{\mathbb{N}}$, the space is of the same cardinality as $\{0,1\}^{\mathbb{N}}$, which is that of the continuum. □

The conclusion can be pushed further: The *Cantor-Bendixon theorem* says that every closed subset of a complete separable metric space decomposes uniquely into the union of a perfect set and a countable set.

Note that, unless we assume the Continuum hypothesis, being at least of cardinality of the continuum is generally more restrictive than just being uncountable (a proof of any perfect subset of $\mathbb{R}$ being uncountable is given in the textbook). However, this is actually not relevant here because, by *Kuratowski's theorem*, every infinite complete and separable metric space is either countable or of the cardinality of the continuum. In short, the Continuum hypothesis actually does hold as a theorem in the set of complete separable metric spaces. (Such spaces are called *Polish*, due to may of these ideas being developed by Polish mathematicians in 1920-30s.)

Interested readers can find further analysis of these questions in textbooks on descriptive set theory as well as book on general topology.