19. Sequential compactness

As part of our proof of completeness of the reals we proved the *Bolzano-Weierstrass theorem* which states that every bounded sequence of the reals contains a convergent subsequence. The aim of this section is to investigate how this concept generalizes to other metric spaces.

19.1 Definition and necessary conditions.

We start by stating the desired property formally:

Definition 19.1 (Sequential compactness) *Let (X,ϱ) be a metric space. A set $A \subseteq X$ is said to be* sequentially compact *if every sequence from A contains a subsequence convergent to a point in A, i.e.,*

$$\forall \{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}\, \exists \{n_k\}_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}\, \exists x \in A\colon n_k \to \infty \wedge x_{n_k} \to x. \tag{19.1}$$

A metric space (X,ϱ) is sequentially compact if the above holds for $A := X$. Here and henceforth $n_k \to \infty$ means $\forall m \in \mathbb{N}\colon \{k \in \mathbb{N}\colon n_k \leqslant m\}$ is finite.

We can check that $A \subseteq X$ is sequentially compact if and only if (A,ϱ_A) is a sequentially compact metric space. This means that, for many statements, we can focus directly on $A := X$. Here is a simple example of a compact space:

Lemma 19.2 (Finite sets are compact) *Any (X,ϱ) with X finite is sequentially compact.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence from X. Writing $z_1,\dots,z_m$ for the points in X, set

$$\forall k = 1,\dots,m\colon \quad I_k := \bigl\{n \in \mathbb{N}\colon x_n = z_k\bigr\}. \tag{19.2}$$

As $\bigcup_{k=1}^m I_k = \mathbb{N}$, there exists $k = 1,\dots,m$ such that I_k is infinite. Let $\{n_j\}_{j\in\mathbb{N}}$ enumerate I_k; i.e., set $n_0 := \inf(I_k)$ and $\forall j \in \mathbb{N}\colon n_{j+1} := \inf\{n \in I_k\colon n > n_j\}$. Then $n_j \geqslant j$ and $x_{n_j} = z_k$ for all $j \in \mathbb{N}$ and so $n_j \to \infty$ and $x_{n_j} \to z_k$ as desired. □

While the previous proof may seems special to the setting of finite sets, the key argument there — which is a version of the "pigeon-hole principle" — is that the union of a finite number of sets is infinite only if one of the sets is infinite. This argument will drive the proofs characterizing sequentially compact sets in $\mathbb{R}^d$ or linking sequential compactness to total boundedness. In this sense, sequentially compact spaces are the closest relatives of finite ones.

We will now observe properties that are implied by, and are thus *necessary* for, sequential compactness. We start by noting:

Lemma 19.3 (AC)(Compactness implies boundedness) *Let (X,ϱ) be a metric space. Then*

$$\forall A \subseteq X\colon \quad A \text{ sequentially compact } \Rightarrow\ A \text{ bounded}. \tag{19.3}$$

Here we recall that a set $A \subseteq X$ is bounded if $\exists x \in X\, \exists r > 0\colon A \subseteq B(x,r)$.

Proof. First we note that the definition of boundedness is equivalent to

$$\forall x \in X\, \exists r > 0\colon A \subseteq B(x,r); \tag{19.4}$$

i.e., with the $\exists$ quantifier for the center of the ball replaced by $\forall$. Indeed, if $A \subseteq B(x_0,r)$ for some $x_0 \in X$ and some $r > 0$, then for each $x \in X$ the triangle inequality shows

$A \subseteq B(x,r')$ where $r' := r + \varrho(x,x_0)$. (Another equivalent way to define boundedness of A is by the boundedness of ϱ_A but this is immaterial for this proof.)

We will now prove (19.3) by proving the contrapositive. The Axiom of Choice will have to be invoked. Suppose that A is NOT bounded. Then

$$\exists x \in X\, \forall n \in \mathbb{N}\colon\ A \smallsetminus B(x,n) \neq \emptyset \tag{19.5}$$

This means that for each $n \in \mathbb{N}$ we may *choose* $x_n \in A \smallsetminus B(x,n)$. Note that then we have $\forall n \in \mathbb{N}\colon\ \varrho(x,x_n) \geqslant n$. Such a sequence $\{x_n\}_{n\in\mathbb{N}}$ cannot contain a convergent (or even Cauchy) subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ because, by Lemma 17.4, that would require that $\{\varrho(x,x_{n_k})\}_{k\in\mathbb{N}}$ be bounded, yet $n_k \geqslant k$ forces $\varrho(x,x_{n_k}) \geqslant k$. Hence, A NOT bounded implies that A is NOT sequentially compact, proving (19.3). □

Lemma 19.4 (AC)(Compactness implies closedness) *Let (X,ϱ) be a metric space. Then*

$$\forall A \subseteq X\colon \quad A \text{ sequentially compact } \Rightarrow\ A \text{ closed.} \tag{19.6}$$

Proof. We again prove the contrapositive. Theorem 16.5 implies

$$\neg(A\ \text{ closed}) \ \Rightarrow\ \exists \{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}\, \exists x \in X\colon x_n \to x\ \wedge\ x \notin A. \tag{19.7}$$

But any subsequence of $\{x_n\}_{n\in\mathbb{N}}$ will then converge to x and so A NOT closed implies A NOT sequentially compact. □

19.2 Cantor diagonal argument.

From the previous lemmas we conclude that boundedness and closedness are necessary conditions for sequential compactness. As it turns out, for $X := \mathbb{R}$ or $\mathbb{R}^d$ endowed with the norm metric, these two conditions are also *sufficient*, thus completely characterizing sequentially compact subsets of the Euclidean space. (We will explain the reasons why the names of E. Heine and E. Borel are attached to this result when we discuss compactness from the topology viewpoint.)

Theorem 19.5 (AC)(Heine-Borel property of $\mathbb{R}^d$) *Let $d \geqslant 1$ be a natural and consider the metric space $(\mathbb{R}^d,\varrho)$ where ϱ is a norm-metric on $\mathbb{R}^d$. Then*

$$\forall A \subseteq \mathbb{R}^k\colon\ A \text{ sequentially compact } \Leftrightarrow\ A \text{ closed and bounded.} \tag{19.8}$$

The AC is used only for the direction $\Rightarrow$.

Proof. We have already proved $\Rightarrow$ (with the help of AC) in the above lemmas, so let us focus on $\Leftarrow$. This direction was in fact already stated in Corollary 17.14 which was proved along with the completeness of the Euclidean spaces in Theorem 17.10. We will nonetheless provide a different reduction to completeness relying on (another instance of) Cantor's diagonal argument. By Proposition 17.11, it suffices to prove the claim just for the ∞-metric ϱ_∞.

Let $A \subseteq \mathbb{R}^d$ be closed and bounded and let $\{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$. By shift and scaling (which do not affect convergence of sequences) we may assume that $A \subseteq Q_0 := [0,1]^d$. The cube Q_0 is covered by 2^d translates $Q_0^{(1)},\dots,Q_0^{(2^d)}$ of $[0,1/2]^d$ one of which must contain (by the aforementioned "pigeon-hole principle") infinitely many terms in the sequence. Denoting $I_0 := \mathbb{N}$ and $J_0^{(i)} := \{n \in I_0 \colon x_n \in Q_0^{(i)}\}$, it is thus meaningful to set

$$i_1 := \min\{i = 1, \dots, 2^d \colon J_0^{(i)} \text{ infinite}\} \tag{19.9}$$

and let $I_1 := J_0^{(i_1)}$ and $Q_1 := Q_0^{(i_1)}$. Then I_1 is infinite and $\forall n \in I_1 \colon x_n \in Q_1$.

We now use the same argument recursively to define a sequence $\{Q_k\}_{k\in\mathbb{N}}$ of cubes and subsets $\{J_k\}_{k\in\mathbb{N}}$ as follows: Assume that for some $k \in \mathbb{N}$, a translate Q_k of $[0, 2^{-k}]$ and an infinite set $I_k \subseteq \mathbb{N}$ are defined such that $\forall n \in I_k \colon x_n \in Q_k$. Then cover Q_k by 2^d translates $Q_k^{(1)}, \dots, Q_k^{(2^d)}$ of $[0, 2^{-(k+1)}]$ so that these have disjoint interiors. Label these in some predetermined fashion (so that no choice of labeling is required in each step). One these cubes must then contain infinitely many terms of the subsequence $\{x_n\}_{n\in I_k}$. Denoting $J_k^{(i)} := \{n \in I_k \colon x_n \in Q_k^{(i)}\}$, we then set

$$i_{k+1} := \min\{i = 1, \dots, 2^d \colon J_k^{(i)} \text{ infinite}\} \tag{19.10}$$

and set

$$I_{k+1} := J_k^{(i_{k+1})} \ \wedge\ Q_{k+1} := Q_k^{(i_{k+1})} \tag{19.11}$$

Proceeding recursively, Q_k and J_k is defined for all $k \in \mathbb{N}$.

The recursive definition ensures

$$\forall k \in \mathbb{N} \colon Q_k \text{ translate of } [0, 2^{-k}]^d \ \wedge\ Q_{k+1} \subseteq Q_k \tag{19.12}$$

and

$$\forall k \in \mathbb{N} \colon\ I_k \text{ infinite } \wedge\ I_{k+1} \subseteq I_k \ \wedge \forall n \in I_k \colon x_n \in Q_k \tag{19.13}$$

Remark 19.6 The set I_k induces a subsequence $\{x_{n_i^{(k)}}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ where $\{n_i^{(k)}\}_{i\in\mathbb{N}}$ is a strictly increasing sequence of naturals enumerating I_k. The condition $I_{k+1} \subseteq I_k$ then shows that the subsequences are nested, meaning that $\{n_i^{(k+1)}\}_{i\in\mathbb{N}}$ is a subsequence of $\{n_i^{(k)}\}_{i\in\mathbb{N}}$. *Cantor's diagonal argument* is a way to choose a subsequence $\{\hat{n}_k\}_{k\in\mathbb{N}}$ simultaneously from all these nested subsequences by the following recipe: Take the k-th term from the k-th subsequence,

$$\forall k \in \mathbb{N} \colon\ \hat{n}_k := n_k^{(k)}. \tag{19.14}$$

Technically, only the part of the *diagonal sequence* $\{\hat{n}_k\}_{k\in\mathbb{N}}$ with indices $k \geqslant \ell$ is a subsequence of $\{n_i^{(\ell)}\}_{i\in\mathbb{N}}$, but this is usually ignored.

Working with subsequences of subsequences is notationally challenging at times and so we will stick with the I_k's. In our notation the above boils down to:

Lemma 19.7 (Cantor's diagonal argument) *Let $\{I_k\}_{k\in\mathbb{N}} \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ be such that*

$$\forall k \in \mathbb{N} \colon I_k \text{ infinite } \wedge\ I_{k+1} \subseteq I_k \tag{19.15}$$

Then

$$n_0 := \inf(I_0) \ \wedge \forall k \in \mathbb{N} \colon\ n_{k+1} := \inf\{n \in I_{k+1} \colon n > n_k\} \tag{19.16}$$

defines a sequence $\{n_k\}_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that

$$\forall k \in \mathbb{N} \colon\ n_k \in I_k \tag{19.17}$$

Proof. This follows from Lemma 9.7 and the fact that the set on the right of (19.16) is infinite, and thus non-empty, for each $k \in \mathbb{N}$. □

We are now ready to conclude: Define $\{n_k\}_{k\in\mathbb{N}}$ by (19.16). Since $n_k \geqslant k$ for all $k \in \mathbb{N}$, (19.13) and (19.17) give

$$\forall k \in \mathbb{N}\colon\ x_{n_k} \in Q_k \tag{19.18}$$

which by the fact that Q_k is a translate of $[0, 2^{-k}]$ shows

$$\forall k, \ell \in \mathbb{N}\colon k \leqslant \ell \ \Rightarrow\ \varrho_\infty(x_k, x_\ell) \leqslant 2^{-k}. \tag{19.19}$$

Lemma 14.7 then implies that $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence. The completeness of $(\mathbb{R}^d, \varrho_\infty)$ proved in Theorem 17.10 then shows that this sequence converges and, since A is closed, the limit belongs to A. Hence, A is sequentially compact as claimed. □

The previous proof relied on the completeness of the underlying space. This is no loss in light of completeness being another sufficient condition for compactness:

Lemma 19.8 (Compactness implies completeness) *Let (X, ϱ) be a metric space. Then*

$$(X, \varrho) \text{ compact } \Rightarrow\ (X, \varrho) \text{ complete.} \tag{19.20}$$

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be Cauchy. If $\{x_n\}_{n\in\mathbb{N}}$ had a convergent subsequence, say $x_{n_j} \to x$, then the Cauchy property would ensure $x_n \to x$ (HW problem) and so every Cauchy sequence would be convergent. Thus compactness implies completeness. □

Notwithstanding, even with completeness in place, a key restriction of the proof was finite-dimensionality of $\mathbb{R}^d$, as seen in the following example: Consider the set of bounded sequences

$$\ell^\infty(\mathbb{N}) := \Bigl\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}\colon \text{bounded}\Bigr\} \tag{19.21}$$

endowed with the metric ϱ_∞ associated with the norm

$$\bigl\|\{x_n\}_{n\in\mathbb{N}}\bigr\|_\infty := \sup_{n\in\mathbb{N}} |x_n| \tag{19.22}$$

where the supremum on the right abbreviates $\sup\{x_n\colon n \in \mathbb{N}\}$. (That this is a norm is checked just as for the ∞-norm on $\mathbb{R}^d$.) The space $(\ell^\infty(\mathbb{N}), \varrho_\infty)$ is also easily checked to be complete. However, it fails to have the Heine-Borel property since the closed unit ball (which is bounded)

$$B'(0,1) := \Bigl\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}\colon \sup_{n\in\mathbb{N}} |x_n| \leqslant 1\Bigr\} \tag{19.23}$$

contains sequences $x^{(k)} := \{x_n^{(k)}\}_{n\in\mathbb{N}}$ defined as

$$x_n^{(k)} := \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{else,} \end{cases} \tag{19.24}$$

whose terms are all unit distance apart, $\|x^{(k)} - x^{(\ell)}\|_\infty = 1$ when $k \geqslant \ell$. Such a sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ cannot contain convergent, or even Cauchy, subsequences.

Note that the latter holds in spite of the Cantor diagonal argument being applicable here: Any sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ of "points" $x^{(k)} := \{x_n^{(k)}\}_{n\in\mathbb{N}} \in B'(0,1)$ has all coordinates contained in $[-1,1]$. Using the same argument as in (19.9–19.17) we can choose a subsequence such that the first coordinate converges, from this one can choose another subsequence such that the second coordinate converges, etc. Along the diagonal sequence (19.14), all coordinates converge. However, and this is the catch or a key point, convergence of coordinates is not sufficient to ensure the convergence in the ∞-norm metric, which requires that coordinates converge *uniformly*.

The problem is actually not specific to the choice of the norm-metric: Closed norm-metric balls in complete linear vector spaces are NOT sequentially compact if (and only if) the space is infinitely dimensional.

19.3 Total boundedness.

In light of the previous counterexample, the question is what other conditions should we add on the right of (19.8) to guarantee sequential compactness. This comes in:

Definition 19.9 (Total boundedness) *We say that a set $A \subseteq X$ is* totally bounded *if*

$$\forall r > 0\, \exists n \in \mathbb{N}\ \exists x_0, \dots, x_n \in A\colon \quad A \subseteq \bigcup_{i=0}^{n} B(x_i, r). \tag{19.25}$$

The space (X, ϱ) is totally bounded if this applies to $A := X$.

Total boundedness implies boundedness, indeed, (19.25) shows that

$$A \subseteq B(x_0, r') \quad \text{for} \quad r' := r + (n+1)\ \max\big\{\varrho(x_0, x_j)\colon i = 0, \dots, n\big\}, \tag{19.26}$$

but the converse is generally false. Also note that we actually do not need to require (19.25) for all $r > 0$; it suffices to require this for a sequence of r's tending to zero. We can also check that the total boundedness is inherited to relative topologies; indeed, if (X, ϱ) is totally bounded, so is every subset $A \subseteq X$ (prove this!). In particular, we only need to prove statements about total boundedness of the whole space.

The reason why we introduce total boundedness is that it is another necessary condition for sequential compactness:

Lemma 19.10 (AC)(Compactness implies total boundedness) *Let (X, ϱ) be a metric space. Then*

$$(X, \varrho) \text{ sequentially compact} \ \Rightarrow\ (X, \varrho) \text{ totally bounded}. \tag{19.27}$$

Proof. We will again aim to prove the contrapositive. Suppose that (X, ϱ) is NOT totally bounded. Then

$$\exists r > 0\ \forall n \in \mathbb{N}\ \forall x_0, \dots, x_n \in X\colon \quad X \smallsetminus \bigcup_{i=0}^{n} B(x_i, r) \neq \emptyset. \tag{19.28}$$

Using the Axiom of Choice, we may thus choose a sequence $\{z_k\}_{k \in \mathbb{N}}$ such that

$$x_0 \in X\ \wedge\ \forall k \in \mathbb{N}\colon x_{k+1} \in X \smallsetminus \bigcup_{i=0}^{k} B(x_i, r) \tag{19.29}$$

Now note that $\varrho(x_i, x_{n+1}) \geqslant r$ for all $i = 0, \dots, n$ and so we have

$$\forall m, n \in \mathbb{N}\colon \quad m \neq n\ \Rightarrow\ \varrho(x_m, x_n) \geqslant r. \tag{19.30}$$

This again implies that $\{x_n\}_{n\in\mathbb{N}}$ contains no convergent subsequence and so (X,ϱ) is NOT sequentially compact. □

The total boundedness offers a way to approximate the metric space (X,ϱ) by a finite metric space — namely, the space $\{x_0,\dots,x_n\}$ consisting of the centers of the r-balls $B(x_0,r),\dots,B(x_n,r)$ that cover X. This allows us to build on the proof of sequential compactness in finite spaces and, particularly, on the argument underlying the characterization of sequential compactness in Euclidean spaces:

Theorem 19.11 (AC) *Let (X,ϱ) be a metric space. Then*

$$(X,\varrho) \text{ sequentially compact} \;\Leftrightarrow\; (X,\varrho) \text{ complete and totally bounded.} \tag{19.31}$$

In particular, if (X,ϱ) is complete then

$$\forall A\subseteq X\colon\; A \text{ sequentially compact} \;\Leftrightarrow\; A \text{ closed and totally bounded} \tag{19.32}$$

Proof. The proof of ⇒ reduces to the above lemmas, so we just need to prove ⇐. We proceed very much like in the proof of Theorem 19.5. First, using the total boundedness of (X,ϱ), for each $k\in\mathbb{N}$ there is $m_k\in\mathbb{N}$ and the points $z_0^{(k)},\dots,z_{m_k}^{(k)}$ such that

$$\bigcup_{i=0}^{m_k} B\big(z_i^{(k)},2^{-k}\big) = X \tag{19.33}$$

(A *choice* of these must be made without further information on X, which requires the Axiom of Choice.) Given $\{x_n\}_{n\in\mathbb{N}}\in X^{\mathbb{N}}$, we now define a sequence $\{I_k\}_{k\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ and a sequence $\{B_k\}_{k\in\mathbb{N}}$ of open balls in X recursively as follows: Since X is bounded, there are $r>0$ and $z\in X$ such that $X=B(z_0,r)$. Then set $I_0:=\mathbb{N}$ and $Q_0:=B(z_0,r_0)$. Next, assume that that for some $k\in\mathbb{N}$ the infinite sets $I_0,\dots,I_k\subseteq\mathbb{N}$ and open balls $B_0,\dots,B_k$ have already been defined, denote

$$\forall i=1,\dots,m_{n+1}\colon\quad J_k^{(i)} := \Big\{j\in I_k\colon x_j\in B\big(z_i^{(k+1)},2^{-(k+1)}\big)\Big\}. \tag{19.34}$$

and, noting that (19.33) implies

$$\bigcup_{i=0}^{m_{k+1}} J_k^{(i)} = I_k \tag{19.35}$$

the "pigeon-hole principle" forces at least one of the $J_k^{(i)}$'s to be infinite. This means that we can set

$$i_{k+1} := \min\big\{i\in\{1,\dots,m_{k+1}\}\colon J_k^{(i)} \text{ infinite}\big\}. \tag{19.36}$$

and let

$$I_{k+1} := J_k^{(i_{k+1})} \;\wedge\; B_{k+1} := B\big(z_{i_{k+1}}^{(k+1)},2^{-(k+1)}\big) \tag{19.37}$$

Since I_{k+1} is infinite, the recursive definition can proceed for all $n\in\mathbb{N}$.

The recursive definition now ensures

$$\forall k\in\mathbb{N}\colon\quad I_k \text{ infinite} \;\wedge\; I_{k+1}\subseteq I_k \;\wedge\; \forall n\in I_k\colon x_n\in B_k. \tag{19.38}$$

The fact that B_k is an open ball of radius 2^{-k} along with the triangle inequality gives

$$\forall k\in\mathbb{N}\,\forall m,n\in I_k\colon \varrho(x_n,x_m) < 2\cdot 2^{-k} = 2^{1-k} \tag{19.39}$$

Define the "diagonal" sequence $\{n_k\}_{k\in\mathbb{N}}$ as in Lemma 19.7. Then (19.17) and $n_k \geqslant k$ give

$$\forall j,k \in \mathbb{N}\colon\, j \leqslant k \;\Rightarrow\; \varrho(x_{n_j}, x_{n_k}) \leqslant 2^{-n_k+1} \leqslant 2^{1-k} \tag{19.40}$$

showing that $\{x_{n_k}\}_{k\in\mathbb{N}}$ is Cauchy. Since (X,ϱ) is assumed complete, $\{x_{n_k}\}_{k\in\mathbb{N}}$ is convergent and so (X,ϱ) is sequentially compact. □

It can be checked that a totally bounded space has a compact completion. Metric spaces that have compact completion are called *precompact*. (In topological spaces that are not metric, the notion of a completion is meaningless, but one then says that a set is precompact if its closure is compact.) The above shows that being precompact is equivalent to being totally bounded. A direct way to define a precompact set is by saying that every sequence drawn from the set has a Cauchy subsequence.