18. Contraction maps and completion

Here we continue discussing completeness albeit now for general metric spaces. For spaces that are not complete, we introduce the notion of their completion. As it turns out, this will give us yet another construction of the reals.

18.1 Completness and its consequences.

Complete metric spaces have a number of attractive properties that makes working with them more convenient. We start by making some general observations about complete spaces. The first one relates completeness to closedness:

Lemma 18.1 (AC)(Inheritance to closed subsets) *Let (X, ϱ) be a complete metric space and, given $A \subseteq X$, let ϱ_A be the metric induced on A. Then for all non-empty $A \subseteq X$,*

$$(A, \varrho_A) \text{ complete} \quad \Leftrightarrow \quad A \text{ closed.} \tag{18.1}$$

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in (X, ϱ) is equivalent to $\{x_n\}_{n\in\mathbb{N}}$ being Cauchy in (A, ϱ_A) so all Cauchy sequences in (A, ϱ_A) converge to some point in X. By Theorem 16.5 (which requires AC), this point is in A for all such sequences if and only if A is closed. □

Another type of inheritance concerns Cartesian products. Here we note that if (X, ϱ_X) and (Y, ϱ_Y) be metric spaces, then $\varrho\colon X \times Y \to \mathbb{R}$ defined by

$$\varrho_\infty\big((x, y), (\tilde{x}, \tilde{y})\big) = \max\big\{\varrho_X(x, \tilde{x}), \varrho_Y(y, \tilde{y})\big\} \tag{18.2}$$

is a metric on $X \times Y$. We write the infinity symbol because the ∞-norm on $\mathbb{R}^2$ is used implicitly to combine the two metrics into one. If instead another norm (e.g., the Euclidean norm) was used, we would get another metric, which by Proposition 17.11 turns out to be equivalent to ϱ_∞ according to the following definition:

Definition 18.2 (Equivalent metrics) *Let ϱ and ϱ' be two metrics on X. We say that ϱ and ϱ' are* equivalent *if*

$$\exists c, C > 0\, \forall x, y \in X\colon \;\; c\varrho(x, y) \leqslant \varrho'(x, y) \leqslant C\varrho(x, y) \tag{18.3}$$

We leave it to the reader to check:

Lemma 18.3 *Equivalent metrics have the same Cauchy and convergent sequences, as well as the same induced topologies.*

We then put forward:

Definition 18.4 *Let (X, ϱ_X) and (Y, ϱ_Y) be metric spaces. The associated* product metric space *is the space $(X \times Y, \varrho)$ where ϱ is any metric equivalent to ϱ_∞ in* (18.2).

We then have:

Lemma 18.5 (Inheritance under Cartesian products) *If (X, ϱ_X) and (Y, ϱ_Y) are complete, then so is the product metric space $(X \times Y, \varrho)$, for any metric ϱ that is equivalent to ϱ_∞ in* (18.2).

Proof. Let ϱ be a metric on $X \times Y$ that obeys $c\varrho(\cdot,\cdot) \leqslant \varrho_\infty(\cdot,\cdot) \leqslant C\varrho(\cdot,\cdot)$. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X \times Y, \varrho_\infty)$. Since

$$\varrho_X(x_n, x_m) \leqslant \varrho_\infty\big((x_n, y_n), (x_m, y_m)\big) \leqslant C\varrho\big((x_n, y_n), (x_m, y_m)\big) \tag{18.4}$$

also $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in X and, by the same argument, $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy in Y. The assumed completeness implies existence of $x \in X$ and $y \in Y$ such that $x_n \to x$ and $y_n \to y$. Then $\varrho_X(x_n, x) \to 0$ and $\varrho_Y(y_n, y) \to 0$, which implies $\varrho_\infty((x_n, y_n), (x, y)) \to 0$ and thus also $\varrho((x_n, y_n), (x, y)) \to 0$. Hence $(x_n, y_n) \to (x, y)$ and $(X \times Y, \varrho)$ is thus complete. □

Having noted that completeness inherits nicely downward and upward, we now move to one important practical consequence of completeness, which is the fact that maps that contract distances admit a fixed point. We start with:

Definition 18.6 (Contraction map) *Let* (X, ϱ) *be a metric space. A map* $\phi\colon X \to X$ *(with* $\mathrm{Dom}(\phi) = X$*) is a* contraction *if*

$$\exists c \in \mathbb{R}\colon \quad 0 \leqslant c < 1 \,\wedge\, \forall x, y \in X\colon\ \varrho\big(\phi(x), \phi(y)\big) \leqslant c\varrho(x, y) \tag{18.5}$$

The fact that $c < 1$ is crucial for this notion. That being said, we warn the reader that the terminology is broken because linear operators (i.e., linear maps of linear spaces) are called contractions if the above holds with $c = 1$. Using the above definition, we have:

Theorem 18.7 (Banach's contraction principle) *Let* (X, ϱ) *be a metric space and let* $\phi\colon X \to X$ *be a contraction map as in* (18.5)*. Then*

$$\exists x \in X\colon\ \phi(x) = x \tag{18.6}$$

meaning that ϕ *admits a fixed point. Moreover, the fixed point is unique,*

$$\forall x, y \in X\colon\ \big(\phi(x) = x \wedge \phi(y) = y\big) \;\Rightarrow\; x = y \tag{18.7}$$

In words, a contraction on a complete metric space has a unique fixed point.

Proof. Let $c \in \mathbb{R}$ and $\phi\colon X \to X$ be a contraction such that (18.5) holds. Pick $x \in X$ and use recursion to construct $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ so that

$$x_0 = x \,\wedge\, \forall n \in \mathbb{N}\colon\ x_{n+1} = \phi(x_n) \tag{18.8}$$

We claim that

$$\forall n \in \mathbb{N}\colon\ \varrho(x_n, x_{n+1}) \leqslant c^n \varrho(x_0, x_1). \tag{18.9}$$

This is proved by induction: Let P_n be the statement after the quantifier. Then P_0 holds trivially because $c^0 = 1$ and if P_n, then the contraction property (18.5) implies

$$\varrho(x_{n+1}, x_{n+2}) = \varrho\big(\phi(x_n), \phi(x_{n+1})\big) \leqslant c\varrho(x_n, x_{n+1}) \overset{P_n}{\leqslant} c \cdot c^n \varrho(x_0, x_1) \tag{18.10}$$

showing $P_n \Rightarrow P_{n+1}$ with the help of $c^{n+1} = c \cdot c^n$. Hereby we get (18.9) via Lemma 4.3.

Next we upgrade (18.9) into

$$\forall n, m \in \mathbb{N}\colon\ n \leqslant m \;\Rightarrow\; \varrho(x_n, x_m) \leqslant \frac{c^n - c^m}{1 - c}\varrho(x_0, x_1). \tag{18.11}$$

We again prove this by induction, this time on m. Let P_m be the logical sentence that the inequality holds for all $n \in \mathbb{N}$ satisfying $n \leqslant m$. The base case P_0 is checked immediately, because then the only non-trivial value is $n = 0$ for which the distance on the left vanishes while the right-hand side is non-negative because $c^m \leqslant c^n$ thanks to $n \leqslant m$. If P_m is TRUE, then for any $n \leqslant m$ (18.9) gives

$$\begin{aligned}\varrho(x_n, x_{m+1}) &\leqslant \varrho(x_n, x_m) + \varrho(x_m, x_{m+1}) \\ &\leqslant \frac{c^n - c^m}{1-c}\varrho(x_0, x_1) + c^m\varrho(x_0, x_1) = \frac{c^n - c^{m+1}}{1-c}\varrho(x_0, x_1).\end{aligned} \tag{18.12}$$

As for case $n = m + 1$ the clause P_{m+1} holds trivially, we get $P_m \Rightarrow P_{m+1}$ and so (18.11) is TRUE as stated by Lemma 4.3.

Dropping the c^m term from the numerator of (18.11) shows

$$\forall n, m \in \mathbb{N}\colon\ n \leqslant m \ \Rightarrow\ \varrho(x_n, x_m) \leqslant \frac{\varrho(x_1, x_0)}{1-c}c^n \tag{18.13}$$

Lemma 14.7 then gives

$$\forall \epsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall n \geqslant n_0\colon\ \frac{\varrho(x_1, x_0)}{1-c}c^n < \epsilon \tag{18.14}$$

proving that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. By the assumed completeness of (X, ϱ), there is $x \in X$ such that $x_n \to x$. Then

$$\begin{aligned}\varrho\bigl(\phi(x), x\bigr) &\leqslant \varrho\bigl(\phi(x), x_{n+1}\bigr) + \varrho(x, x_{n+1}) \\ &= \varrho\bigl(\phi(x), \phi(x_n)\bigr) + \varrho(x, x_{n+1}) \leqslant c\varrho(x, x_n) + \varrho(x, x_{n+1})\end{aligned} \tag{18.15}$$

and since both terms on the right tend to zero, we get $\varrho(\phi(x), x) = 0$ implying $\phi(x) = x$ as desired. The fixed point is unique because if x and y are both fixed points, then $\varrho(x, y) = \varrho(\phi(x), \phi(y)) \leqslant c\varrho(x, y)$ which forces $\varrho(x, y) = 0$ and thus $x = y$. □

Another name for Theorem 18.7 is *Banach's fixed point theorem*. While we will not give applications of the above theorem at this time, we note that Theorem 18.7 finds many practical uses most of which, however, are phrased using terms (such as the space of continuous functions) that we do not yet have the tools to discuss here.

Note also that the proof actually suggests an algorithm for constructing the fixed point: Iterate the map successively starting from an arbitrary point. This is in fact how this method is often used in practice; for instance, when constructing solutions of differential equations by way of so called *Picard iterations*.

18.2 Intrinsic closedness.

Let us now move to more abstract aspects of completness. As noted in Lemma 18.1, completeness is somehow analogous to (sequential characterization of) closedness, albeit with convergent sequences replaced by Cauchy sequences. We will now expound on this connection further. We need:

Definition 18.8 (Isometry) *Let (X, ϱ_X) and (Y, ϱ_Y) be metric spaces. We say that the map $\phi\colon X \to Y$ is an* isometry *or an* isometric embedding of X into Y *if*

$$\forall x, y \in X\colon\quad \varrho_Y\bigl(\phi(x), \phi(y)\bigr) = \varrho_X(x, y). \tag{18.16}$$

We will always assume that the isometry has full domain; i.e., $\mathrm{Dom}(\phi) = X$.

Thus, unlike contractions that shrink distances, isometries preserve them. The latter property implies:

Lemma 18.9 *Any isometry is automatically injective.*

Proof. Let $x, y \in A$ and suppose that $\phi(y) = \phi(x)$. Then (18.16) implies $\varrho_X(x,y) = 0$ which by the separation axiom for the metric gives $x = y$. □

Not all isometries are necessarily onto, of course. For instance, $(\mathbb{R}, \varrho)$ with $\varrho(x,y) := |x-y|$ embeds isometrically into $(\mathbb{R}^d, \varrho_p)$ for any $p \in [1,\infty]$ yet the embedding is not surjective. We thus introduce another qualifier:

Definition 18.10 *An isometric isomorphism (a.k.a. bijective isometry) is an isometry which is onto.*

Two spaces related by an isometric isomorphism are indistinguishable as far as their metric properties are concerned. Relating two metric spaces by an isomorphism is thus saying that they are basically the same. We now characterize complete metric spaces by closedness of their isometric embedding in other complete spaces:

Theorem 18.11 (Intrinsic closedness of complete spaces) *Let (X,ϱ) be a metric space. Then the following are equivalent:*

(1) (X,ϱ) *is complete*
(2) $\forall (Y,\varrho')$ complete $\forall \phi\colon X \to Y$ isometry: $\phi(X)$ is closed in (Y,ϱ')

In words, a space is complete if and only if its embedding into any complete space is closed.

Proof of (1) $\Rightarrow$ (2). Assume that (X,ϱ) and (Y,ϱ') are complete and let $\phi\colon X \to Y$ be an isometry. Consider a sequence $\{y_n\}_{n\in\mathbb{N}} \in \phi(X)^{\mathbb{N}}$ such that $y_n \to y$, for some $y \in Y$. Then there is $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ such that $\phi(x_n) = y_n$ for each $n \in \mathbb{N}$. The assumed convergence implies that $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy and, using that ϕ is an isometry we readily check that so is $\{x_n\}_{n\in\mathbb{N}}$. By completeness of (X,ϱ) there exists $x \in X$ such that $x_n \to x$. The isometry property now shows that $y_n = \phi(x_n) \to \phi(x)$. The uniqueness of the limit then gives $y = \phi(x)$, implying $y \in \phi(X)$. By Theorem 16.5, $\phi(X)$ is closed. □

The term *intrinsic* has been used to emphasize that a complete space embeds isometrically to any complete space (for which such an embedding exists) as a closed set. That (1) and (2) are equivalent means that for incomplete spaces this fails for *all* isometric embeddings (not just one).

The proof of the opposite implication is harder as it requires the introduction (and construction of an instance) of the following concept:

Definition 18.12 (Completion) *Let (X,ϱ) be a metric space. A* completion of X *is any metric space $(\overline{X},\overline{\varrho})$ such that*

(1) $(\overline{X},\overline{\varrho})$ *is complete, and*
(2) $\exists \phi\colon X \to \overline{X}$ *isometry such that* $\overline{\phi(X)} = \overline{X}$.

Here, in (2), the closure of $\phi(X)$ is in the metric space $(\overline{X},\overline{\varrho})$.

A few remarks are in order:

- The notation $\overline{X}$ has, *a priori*, nothing to do with closure; it is just a notation for the completion. However, in light of (2), it is in fact a sort of a closure as (2) says that there is an embedding of X is which the closure of X is all of $\overline{X}$.
- Condition (2) is a minimality condition. Indeed, we already noted that $(\mathbb{Q}, \varrho)$ can be embedded into $(\mathbb{R}, \varrho)$, which is complete and is in fact the closure of $(\mathbb{Q}, \varrho)$, but also into $(\mathbb{R}^d, \varrho_p)$ for any $d \geqslant 1$ and any $p \in [1, \infty]$. We would not want to regard the latter spaces as the completion of $(\mathbb{Q}, \varrho)$.
- Using an earlier definition, condition in (2) means that $\phi(X)$ is dense in $\overline{X}$.

We now claim:

Theorem 18.13 *For each metric space there is at least one completion.*

Leaving the proof to the next subsection, we note that this is enough to give the proof of the opposite implication in Theorem 18.11:

Proof of (2) $\Rightarrow$ (1) *in Theorem 18.11.* Consider the complete space $(Y, \varrho') := (\overline{X}, \overline{\varrho})$ and let ϕ be the isometric embedding of X into $\overline{X}$. If $\phi(X)$ is closed then $\phi(X) = \overline{X}$ and ϕ is thus onto. Then X is isometric to a complete space and so it is thus complete. □

18.3 Existence of a completion.

We now move to the proof of Theorem 18.13. The argument builds on Cantor's 1878 proof of existence of a system reals which was based on the fact that one way to think of a real number as a convergent sequence of rationals.

While such a representation is quite natural, a number of conceptual problems arise in its rigorous implementation. The first one is that many sequences of rationals converge to the same real number. We thus somehow need to find a way to identify the sequences with the same limit as one. This will be done by grouping them into equivalence classes under a suitable equivalence relation. Another problem is that, before the reals are actually constructed, some Cauchy sequences of rationals may not converge because there is no limit point for them to converge to. Instead of convergent sequences, we should thus rather focus on Cauchy sequences.

We now move to implementing this strategy in the context of a general metric space (X, ϱ). We start with by grouping Cauchy sequences in equivalence classes. Given two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, we set

$$\{x_n\}_{n\in\mathbb{N}} \sim \{y_n\}_{n\in\mathbb{N}} := \lim_{n\to\infty} \varrho(x_n, y_n) = 0 \tag{18.17}$$

We leave it to the reader to check that this is a reflexive, symmetric and transitive relation on the set of Cauchy sequences, and thus is an equivalence relation. The equivalence class of associated with a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ is then

$$\bigl[\{x_n\}_{n\in\mathbb{N}}\bigr] := \Bigl\{\{y_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}} \colon \lim_{n\to\infty} \varrho(x_n, y_n) = 0\Bigr\}. \tag{18.18}$$

We leave it to the reader to check the easy consequences of above definitions:

Lemma 18.14 *For any Cauchy sequence* $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$,

(1) $\{x_n\}_{n\in\mathbb{N}} \in \bigl[\{x_n\}_{n\in\mathbb{N}}\bigr]$,
(2) $\forall \{y_n\}_{n\in\mathbb{N}} \in \bigl[\{x_n\}_{n\in\mathbb{N}}\bigr] \colon \{y_n\}_{n\in\mathbb{N}}$ *is Cauchy,*

(3) $\forall \{y_n\}_{n\in\mathbb{N}}, \{\tilde{y}_n\}_{n\in\mathbb{N}} \in \bigl[\{x_n\}_{n\in\mathbb{N}}\bigr]\colon \varrho(y_n,\tilde{y}_n) \to 0$ *and so*

$$\forall \{y_n\}_{n\in\mathbb{N}} \in \bigl[\{x_n\}_{n\in\mathbb{N}}\bigr]\colon\ \bigl[\{y_n\}_{n\in\mathbb{N}}\bigr] = \bigl[\{x_n\}_{n\in\mathbb{N}}\bigr] \tag{18.19}$$

With these in hand, we set

$$\overline{X} := \Bigl\{[\{x_n\}_{n\in\mathbb{N}}] \in X^{\mathbb{N}}\colon\ \{x_n\}_{n\in\mathbb{N}} \text{ Cauchy}\Bigr\}. \tag{18.20}$$

Our next goal is to define a metric on $\overline{X}$. To this end, we recall a lemma based on an exercise from homeowork:

Lemma 18.15 *For any two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}, \{\tilde{x}_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$,*

$$\{\varrho(x_n,\tilde{x}_n)\}_{n\in\mathbb{N}} \text{ is Cauchy} \tag{18.21}$$

and so

$$\lim_{n\to\infty} \varrho(x_n,\tilde{x}_n) \text{ exists in } \mathbb{R} \tag{18.22}$$

Moreover, for all $\{y_n\}_{n\in\mathbb{N}} \in \bigl[\{x_n\}_{n\in\mathbb{N}}\bigr]$ and all $\{\tilde{y}_n\}_{n\in\mathbb{N}} \in \bigl[\{\tilde{x}_n\}_{n\in\mathbb{N}}\bigr]$,

$$\lim_{n\to\infty} \varrho(y_n,\tilde{y}_n) = \lim_{n\to\infty} \varrho(x_n,\tilde{x}_n) \tag{18.23}$$

and so the limit depends only on the equivalence classes of the sequences.

Proof. In order to get (18.21), let $n, m \in \mathbb{N}$ and note that

$$\bigl|\varrho(x_m,\tilde{x}_m) - \varrho(x_n,\tilde{x}_n)\bigr| \leqslant \varrho(x_m,x_n) + \varrho(\tilde{x}_m,\tilde{x}_n) \tag{18.24}$$

Thanks to the Cauchy property of the sequences, given $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that both terms on the right-hand side are smaller than $\epsilon/2$ once $n, m \geqslant n_0$. It follows that $\{\varrho(x_n,\tilde{x}_n)\}_{n\in\mathbb{N}}$ is Cauchy. The completeness of $\mathbb{R}$ proved in Theorem 17.2 then gives (18.22). The argument for (18.23) is based on a similar inequality as (18.24) and observation (3) in Lemma 18.14 so we leave it to the reader. □

We can now define $\overline{\varrho}\colon \overline{X} \times \overline{X} \to \mathbb{R}$ by

$$\overline{\varrho}\bigl([\{x_n\}_{n\in\mathbb{N}}], [\{\tilde{x}_n\}_{n\in\mathbb{N}}]\bigr) := \lim_{n\to\infty} \varrho(x_n,\tilde{x}_n) \tag{18.25}$$

where, by (18.23), the limit is independent of the representatives. We then quickly check:

Lemma 18.16 *$\overline{\varrho}$ is a metric on $\overline{X}$.*

Proof. The symmetry and non-negativity are immediate from the corresponding properties of ϱ and so is the triangle inequality. The definition (18.25) along with (18.18) ensure

$$\overline{\varrho}([\{x_n\}_{n\in\mathbb{N}}], [\{\tilde{x}_n\}_{n\in\mathbb{N}}]) = 0 \quad \Rightarrow \{\tilde{x}_n\}_{n\in\mathbb{N}} \in [\{x_n\}_{n\in\mathbb{N}}] \tag{18.26}$$

and so $[\{x_n\}_{n\in\mathbb{N}}] = [\{\tilde{x}_n\}_{n\in\mathbb{N}}]$ by Lemma 18.14(3). □

We are now ready to give:

Proof of Theorem 18.13. Let $(\overline{X}, \overline{\varrho})$ be as above and let $\phi\colon X \to \overline{X}$ be defined by

$$\phi(x) := [\{x\}_{n\in\mathbb{N}}] \tag{18.27}$$

where $\{x\}_{n\in\mathbb{N}}$ denotes the constant sequence whose all terms are equal to x. (This sequence is trivially Cauchy.) The proof now splits into three claims:

Claim 1: ϕ is an isometry. This is immediate from

$$\overline{\varrho}\big([\{x\}_{n\in\mathbb{N}}],[\{\tilde{x}\}_{n\in\mathbb{N}}]\big) = \lim_{n\to\infty}\varrho(x,\tilde{x}) = \varrho(x,\tilde{x}). \tag{18.28}$$

Claim 2: $\phi(X)$ is dense in $\overline{X}$: Pick any $[\{x_n\}_{n\in\mathbb{N}}]\in\overline{X}$. Then

$$\lim_{m\to\infty}\overline{\varrho}\big(\phi(x_m),[\{x_n\}_{n\in\mathbb{N}}]\big) = \lim_{n\to\infty}\varrho(x_m,x_n) = 0 \tag{18.29}$$

where the last conclusion follows from the fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. This implies $\phi(x_m)\to[\{x_n\}_{n\in\mathbb{N}}]$ in $(\overline{X},\overline{\varrho})$ and so $[\{x_n\}_{n\in\mathbb{N}}]$ is an adherent point of $\phi(X)$. As this is true for any $[\{x_n\}_{n\in\mathbb{N}}]\in\overline{X}$, the closure of $\phi(X)$ is all of $\overline{X}$.

Claim 3: $(\overline{X},\overline{\varrho})$ is complete: Consider a sequence $\{[\{x_n^{(m)}\}_{n\in\mathbb{N}}]\}_{m\in\mathbb{N}}$ (indexed by m) of elements in $\overline{X}$ and assume that it is Cauchy in metric $\overline{\varrho}$. As $\phi(X)$ is already known to be dense in $\overline{X}$, for each $m\in\mathbb{N}$ there is $y_m\in X$ such that

$$\overline{\varrho}\big([\{x_n^{(m)}\}_{n\in\mathbb{N}}],\phi(y_m)\big)\leqslant\frac{1}{m+1} \tag{18.30}$$

where, we racall, $\phi(y_m)$ is the equivalence class of Cauchy sequences represented by a constant sequence equal to y_m. Since ϕ is an isometry, the triangle inequality and (18.30) show

$$\forall m,k\in\mathbb{N}:\quad \varrho(y_m,y_k)=\overline{\varrho}\big(\phi(y_m),\phi(y_k)\big)\leqslant\frac{1}{m+1}+\frac{1}{k+1} \tag{18.31}$$

and so $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy. This means that $[\{y_n\}_{n\in\mathbb{N}}]$ is an element of $\overline{X}$. Taking a limit $k\to\infty$ in (18.31) then shows

$$\forall m\in\mathbb{N}:\quad \overline{\varrho}\big(\phi(y_m),[\{y_n\}_{n\in\mathbb{N}}]\big)\leqslant\frac{1}{m+1}. \tag{18.32}$$

Combining (18.30) and (18.32) using the triangle inequality now shows

$$\forall m\in\mathbb{N}:\quad \overline{\varrho}\big([\{x_n^{(m)}\}_{n\in\mathbb{N}}],[\{y_n\}_{n\in\mathbb{N}}]\big)\leqslant\frac{2}{m+1} \tag{18.33}$$

and so $[\{x_n^{(m)}\}_{n\in\mathbb{N}}]\to[\{y_n\}_{n\in\mathbb{N}}]$ in $(\overline{X},\overline{\varrho})$. This proves that $(\overline{X},\overline{\varrho})$ is complete. □

We remark that parts of the above proof can indeed be used (as Cantor did in his paper from 1878) to construct a system of reals out of a system of rationals. Indeed, specialize to $X:=\mathbb{Q}$ and $\varrho(a,b):=|a-b|$ (which takes rational values) and observe that the notion of being Cauchy can be defined using rationals alone (see, e.g., Definition 14.3). Then let $\mathbb{R}$ as the set of classes of equivalence of Cauchy sequences. The map (18.27) then gives us an injection $\mathbb{Q}\to\mathbb{R}$. We now define the algebraic operations on $\mathbb{R}$ as follows

$$\begin{aligned}[\{a_n\}_{n\in\mathbb{N}}]+[\{b_n\}_{n\in\mathbb{N}}] &:= [\{a_n+b_n\}_{n\in\mathbb{N}}]\\ [\{a_n\}_{n\in\mathbb{N}}]\cdot[\{b_n\}_{n\in\mathbb{N}}] &:= [\{a_n\cdot b_n\}_{n\in\mathbb{N}}]\end{aligned} \tag{18.34}$$

(which requires showing that $\{a_n+b_n\}_{n\in\mathbb{N}}$ and $\{a_n\cdot b_n\}_{n\in\mathbb{N}}$ are Cauchy if $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are). Along with $\underline{0}:=\{0\}_{n\in\mathbb{N}}$, $\underline{1}:=\{1\}_{n\in\mathbb{N}}$ and the relation

$$[\{a_n\}_{n\in\mathbb{N}}]\leqslant[\{b_n\}_{n\in\mathbb{N}}] \;:=\; \forall k\in\mathbb{N}\colon \Big\{n\in\mathbb{N}\colon\, a_n\leqslant b_n+\frac{1}{k+1}\Big\}\text{ is infinite} \tag{18.35}$$

we then check that $(\mathbb{R}, +, \underline{0}, \cdot, \underline{1}, \leqslant)$ is an ordered field. A variant of the argument in Claim 3 above then shows that this ordered field is in fact complete and is thus a system of reals. (Of course, all arguments there have to be phrased using rationals only.)

18.4 Uniqueness of the completion.

As our final item we also address the uniqueness of the completion. Of course, this can only be true modulo an isometric bijection:

Theorem 18.17 (Uniqueness up to an isomorphism) *If $(\overline{X}_1, \overline{\varrho}_1)$ and $(\overline{X}_2, \overline{\varrho}_2)$ are two completions of a metric space (X, ϱ), then there is bijection $\overline{\psi}\colon \overline{X}_1 \to \overline{X}_2$ which is an isometry.*

Proof. The definition of a closure ensures existence of the isometries $\phi_i\colon X \to \overline{X}_i$, $i = 1, 2$, such that the closure of $\phi_i(X)$ in $\overline{X}_i$ is all of $\overline{X}_i$. By Lemma 18.9 these maps are injective and so we may define $\psi\colon \phi_1(X) \to \phi_2(X)$ by

$$\psi(x) := \phi_2 \circ \phi_1^{-1}(x). \tag{18.36}$$

Then the fact that both ϕ_1 and ϕ_2 are isometries imply, for all $x, y \in \phi_1(X)$,

$$\begin{aligned} \varrho_2\big(\psi(x), \psi(y)\big) &= \varrho_2\big(\phi_2 \circ \phi_1^{-1}(y), \phi_2 \circ \phi_1^{-1}(x)\big) \\ &= \varrho(\phi_1^{-1}(y), \phi^{-1}(x) = \varrho_1(y, x) \end{aligned} \tag{18.37}$$

and so ψ is an isometry of $\phi_1(X)$ onto $\phi_2(X)$. Consider now any $x \in \overline{X}_1$. Since the closure of $\phi_1(X)$ is $\overline{X}_1$, Corollary 16.6 implies existence of $\{x_n\}_{n\in\mathbb{N}} \in \phi_1(X)^{\mathbb{N}}$ such that $x_n \to x$. Using that ψ is an isometry, we readily check the following facts:

$$x \in \phi_1(X) \;\Rightarrow\; \psi(x_n) \to \psi(x) \tag{18.38}$$

and if $x \notin \phi_1(X)$, then $\psi(x_n)$ is convergent. We may thus define

$$\overline{\psi}(x) := \lim_{n\to\infty} \psi(x_n). \tag{18.39}$$

A small technical caveat is that Corollary 16.6 requires a *choice* of the sequence convergent to x. However, one can check that any sequence that converges to x will lead to the same value of the limit of $\psi(x_n)$.

Noting that $\overline{\psi}$ is defined on all of $\overline{X}_1$, all that remains to prove two claims:

Claim 1: $\overline{\psi}$ is an isometry. Note that if $\{x_n\}_{n\in\mathbb{N}}, \{\tilde{x}_n\}_{n\in\mathbb{N}} \in \phi_1(X)^{\mathbb{N}}$ are such that $x_n \to x$ and $\tilde{x}_n \to \tilde{x}$, then the triangle inequality and the fact that ψ is an isometry on $\phi_1(X)$ shows

$$\Big|\varrho_2\big(\overline{\psi}(x), \overline{\psi}(\tilde{x})\big) - \varrho_1(x, \tilde{x})\Big| \leqslant \varrho_2\big(\overline{\psi}(x), \psi(x_n)\big) + \varrho_2(\psi(\tilde{x}_n), \overline{\psi}(\tilde{x})) \tag{18.40}$$

where the right-hand side tends to zero by (18.39). Hence we get

$$\varrho_2\big(\overline{\psi}(x), \overline{\psi}(\tilde{x})\big) = \varrho_1(x, \tilde{x}) \tag{18.41}$$

and so $\overline{\psi}$ is an isometry as desired.

Claim 2: $\overline{\psi}$ is onto. Let $y \in \overline{X}_2$ and note that, since the closure of $\phi_2(X)$ is all of $\overline{X}_2$, Corollary 16.6 ensures the existence of $\{y_n\}_{n\in\mathbb{N}} \subseteq \phi_2(X)$ such that $y_n \to y$. Since ψ is onto $\phi_2(X)$, for each $n \in \mathbb{N}$ there is $x_n \in X$ be such that $\psi(x_n) = y_n$ and the fact that ψ is an isometry implies that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. As $\overline{X}_1$ is complete, there is $x \in \overline{X}_1$ such that

$x_n \to x$. But the aforementioned independence of (18.39) on the sequence approaching x, we have that $\psi(x_n) \to \overline{\psi}(x)$ which implies $y = \overline{\psi}(x)$. The map $\overline{\psi}$ is thus onto. □

The above can thus be considered as an variation on the proof of the existence and uniqueness of the reals. However, unlike Dedekind's approach, the advantage of the metric-space based approach is its seamless extension to other contexts, and in particular, to linear vector spaces of infinite dimension. This is quite appreciated in the subject of mathematics called functional analysis that deals with such spaces systematically.