17. Completeness

Having discussed the connections between sequences and topology, we now turn back to sequences alone and examine the following basic question: In what metric spaces do all Cauchy sequences converge? We first answer this question for the reals endowed with the Euclidean metric and then treat general metric spaces.

17.1 Completeness of the reals.

As noted earlier, a special name is reserved for the metric spaces for which the above question is answered affirmatively:

Definition 17.1 (Completeness) *We say that a metric space (X, ϱ) is* complete *if every Cauchy sequence is convergent, i.e., if*

$$\forall \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}\colon\ \{x_n\}_{n\in\mathbb{N}} \text{ Cauchy } \Rightarrow\ \exists x \in X\colon x_n \to x. \tag{17.1}$$

If the space is not complete, then we say it is incomplete.

Note that our earlier use of the term "complete" concerned the validity of the least-upper bound property in ordered fields. This is no loss in light of:

Theorem 17.2 *The metric space $(\mathbb{R}, \varrho)$, where $\varrho(x, y) := |x - y|$, is complete.*

The proof will require some facts about convergence of sequences which we will be useful throughout the rest of the course. The first lemma works for all metric spaces and is based on the following concept:

Definition 17.3 *Let (X, ϱ) be a metric space. A set $A \subseteq X$ is said to be* bounded *if it is contained in an open ball, i.e.,*

$$\exists x \in X\, \exists r > 0\colon\ A \subseteq B(x, r) \tag{17.2}$$

If a set is not bounded, then we call it unbounded.

We now observe:

Lemma 17.4 *Let (X, ϱ) be a metric space. Then*

$$\forall \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}\colon\ \{x_n\}_{n\in\mathbb{N}} \text{ Cauchy } \Rightarrow \{x_n\colon n \in \mathbb{N}\} \text{ bounded} \tag{17.3}$$

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a Cauchy sequence. Then (choosing $\epsilon := 1$ in Definition 14.12) there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0\colon\ \varrho(x_n, x_{n_0}) < 1 \tag{17.4}$$

Fix any $x \in X$. The triangle inequality then implies $\varrho(x, x_n) \leqslant \varrho(x, x_{n_0}) + 1$ for all $n \geqslant n_0$ and so we have

$$\forall n \in \mathbb{N}\colon\ \varrho(x, x_n) \leqslant \max\bigl\{\varrho(x, x_k)\colon k \in \mathbb{N} \wedge k \leqslant n_0\bigr\} + 1 \tag{17.5}$$

Denoting the number on the right as $\tilde{r}$, we thus have $\forall n \in \mathbb{N}\colon x_n \in B(x, \tilde{r} + 1)$, proving (17.3) with $r := \tilde{r} + 1$. □

Focusing now on sequences of reals, the next lemma calls upon the notions of "non-decreasing" and "strictly increasing" sequences from Definition 14.2:

Lemma 17.5 *Let $(A, \leqslant)$ be a totally ordered set. For each sequence $\{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that*

$$\forall k \in \mathbb{N}\colon x_{n_k} \leqslant x_{n_{k+1}} \;\vee\; \forall k \in \mathbb{N}\colon x_{n_{k+1}} < x_{n_k} \tag{17.6}$$

In words, each sequence in a totally ordered set contains a subsequence that is either non-decreasing or strictly decreasing.

Proof. Given $\{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ let $J := \{n \in \mathbb{N}\colon (\forall j > n\colon x_j < x_n)\}$, where we set $a < b := a \leqslant b \wedge a \neq b$. If J is finite (empty or non-empty), then $\sup(J)$ exists (and equals 0 when $J = \emptyset$) and belongs to $\mathbb{N}$. Then we recursively define

$$n_0 := \sup(J) + 1 \;\wedge\; \forall k \in \mathbb{N}\colon n_{k+1} := \inf\{j > n_k\colon x_j \geqslant x_{n_k}\}, \tag{17.7}$$

where we note that $n_k \geqslant n_0$ by construction and so $\{j > n_k\colon x_j \geqslant x_{n_k}\} \neq \emptyset$ by the fact that $n_k \notin J$ as implied by $n_k \geqslant n_0 > \sup(J)$. Since n_{k+1} belongs to the set under infimum, we get $x_{n_{k+1}} \geqslant x_{n_k}$ for all $k \in \mathbb{N}$ thus proving the first alternative in (17.6).

If on the other hand J is infinite, then we set

$$n_0 := 0 \;\wedge\; \forall k \in \mathbb{N}\colon\; n_{k+1} := \inf\{j > n_k\colon x_j < x_{n_k}\} \tag{17.8}$$

where the infimum exists and belongs to the set on the right by Lemma 9.7 because that set is infinite for each $k \in \mathbb{N}$. This now readily gives $n_{k+1} > n_k$ and $x_{n_{k+1}} < x_{n_k}$ for all $k \in \mathbb{N}$, proving the second alternative in (17.6). □

Next we call on another important fact about monotone sequences of reals:

Lemma 17.6 (Bounded monotone sequence of reals converge) *Let $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be non-decreasing and bounded from above, i.e.,*

$$\exists c \in \mathbb{R}\, \forall n \in \mathbb{N}\colon\; x_n \leqslant x_{n+1} \;\wedge\; x_n \leqslant c \tag{17.9}$$

Then $\{x_n\}_{n\in\mathbb{N}}$ is convergent and, in fact,

$$\lim_{n\to\infty} x_n = \sup\{x_n\colon n \in \mathbb{N}\} \tag{17.10}$$

If $\{x_n\}_{n\in\mathbb{N}}$ is instead non-increasing (and bounded), then $\lim_{n\to\infty} x_n = \inf\{x_k\colon k \in \mathbb{N}\}$.

Proof. The assumptions (along with the least-upper bound property of $\mathbb{R}$) ensure that the supremum exists. Denote the supremum by c and let $\epsilon \in \mathbb{R}$ obey $\epsilon > 0$. Then $c - \epsilon$ is not an upper bound and so $\exists n_0 \in \mathbb{N}\colon c - \epsilon < x_{n_0}$. But then the monotonicity claim in (17.9) guarantees

$$\forall n \geqslant n_0\colon\; c - \epsilon < x_{n_0} \leqslant x_n \leqslant c < c + \epsilon \tag{17.11}$$

Noting that the extreme ends of these inequalities imply $|x_n - c| < \epsilon$, we have verified (14.5) for all $\epsilon > 0$ and thus proved (17.10). □

With the above lemmas in hand, we have proved a classical result discovered in 1817 by B. Bolzano in his proof of the Intermediate Value Theorem and flagged some 50 years later by K. Weierstrass as a result of independent interest:

Theorem 17.7 (Bolzano-Weierstrass theorem) *Every bounded sequence of reals contains a convergent subsequence.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be a bounded sequence. By Lemma 17.5, there exists strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ is monotone. Being still bounded, this subsequence converges by Lemma 17.6. □

With these in hand, we are ready to give:

Proof of Theorem 17.2. Let $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be a Cauchy sequence. The sequence is then bounded by Lemma 17.4 and so it contains a convergent subsequence by Theorem 17.7. To conclude the claim, we thus need:

Lemma 17.8 *Let (X,ϱ) be a metric space and $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ a Cauchy sequence. If $\{x_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence, then $\{x_n\}_{n\in\mathbb{N}}$ is itself convergent.*

The proof of this lemma is a homework exercise. □

The above demonstrates that the completeness property of the reals as an ordered field is essential for the completeness in the sense of metric spaces. The converse is actually true as well:

Lemma 17.9 *Let F be an ordered subfield of $\mathbb{R}$ which we regard as a metric space (F,ϱ) for the Euclidean metric $\varrho(x,y) = |x-y|$. Then*

$$(F,\varrho) \text{ complete} \quad \Leftrightarrow \quad F \text{ has least upper bound property} \tag{17.12}$$

In particular, no proper ordered subfield of $\mathbb{R}$ is complete in the Euclidean metric.

Proof. For "$\Leftarrow$" in (17.12), recall that every ordered field with least upper bound property is isomorphic to the reals. That $(\mathbb{R},\varrho)$ is complete as a metric space was shown in Theorem 17.2. The implication "$\Rightarrow$" is left to a homework exercise. □

We remark that, in the previous lemma, the restriction to a subfield of the reals is necessary for the metric to take values in $\mathbb{R}$. (As noted in Section 10.5, there are ordered fields larger than $\mathbb{R}$ but in these the absolute value is not generally $\mathbb{R}$-valued.)

We also note that the arguments underpinning Theorem 17.2 depend crucially on the choice of the metric. And, indeed, as noted earlier, all convergent sequences for the reals with the Euclidean metric ϱ also converge in the metric ϱ' in (14.14), but the latter also admits $x_n := n$ as a non-convergent Cauchy sequence. So while $(\mathbb{R},\varrho)$ is complete, $(\mathbb{R},\varrho')$ is not. As noted in homework, this holds regardless of the fact that both metrics induce the same topology.

17.2 Completeness of Euclidean spaces.

There are many complete metric spaces. For instance, Lemma 14.15 shows that every discrete metric space is complete. Our interest is of course in metric space that are pertinent to analysis so our next step is the completeness of the Euclidean spaces.

Theorem 17.10 *Let $d \geqslant 1$ be a natural and let ϱ be a norm-metric on $\mathbb{R}^d$; i.e., $\varrho(x,y) := \|x-y\|$ for $\|\cdot\|$ a norm on $\mathbb{R}^d$. Then $(\mathbb{R}^d,\varrho)$ is complete.*

We start by a useful fact from linear algebra which is proved by analysis.

Proposition 17.11 *Let $d \geqslant 1$ be a natural and let $\|\cdot\|$ be a norm on on $\mathbb{R}^d$. Then*

$$\exists c, C > 0\, \forall x \in \mathbb{R}^d\colon\ c\|x\|_\infty \leqslant \|x\| \leqslant C\|x\|_\infty \tag{17.13}$$

In short, all norms on $\mathbb{R}^d$ are comparable.

Proof. The upper bound in (17.13) is immediate: Writing $x = (x_1, \dots, x_d) = \sum_{i=1}^d x_k e_i$, where $e_1, \dots, e_d$ are the coordinate vectors in $\mathbb{R}^d$, the triangle inequality shows

$$\|x\| \leqslant \sum_{i=1}^{n} |x_i| \|e_1\| \leqslant \Bigl(\sum_{i=1}^{n} \|e_i\|\Bigr) \|x\|_\infty \tag{17.14}$$

so the upper bound in (17.13) holds with $C := \sum_{i=1}^n \|e_i\|$.

We will prove the lower bound with

$$c := \inf\{\|x\| \colon x \in \mathbb{R}^d \wedge \|x\|_\infty = 1\} \tag{17.15}$$

where the infimum exists because the set of reals on the right-hand side is non-empty and bounded below by zero. To give justice to the statement, we need to show that $c > 0$ so let us assume for the sake of contradiction that $c = 0$. Then there exists a sequence $\{x^{(n)}\}_{n\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}}$ with

$$\bigl(\forall n \in \mathbb{N} \colon \|x^{(n)}\|_\infty = 1\bigr) \wedge \|x^{(n)}\| \to 0. \tag{17.16}$$

Writing $x^{(n)} = (x_1^{(n)}, \dots, x_d^{(n)})$, the condition $\|x^{(n)}\|_\infty = 1$ shows that that the coordinate sequences $\{x_i^{(n)}\}_{n\in\mathbb{N}}$ are all bounded, i.e.,

$$\forall n \in \mathbb{N}\, \forall i = 1, \dots, d \colon |x_i^{(n)}| \leqslant 1. \tag{17.17}$$

Invoking the Bolzano-Weierstrass theorem, there exists a strictly increasing subsequence $\{n_k^{(1)}\}_{k\in\mathbb{N}}$ such that $\{x_i^{(n_k^{(1)})}\}_{k\in\mathbb{N}}$ is convergent. By induction we then prove that, for each $m = 2, \dots, d$, there exists a strictly increasing sequence $\{n_k^{(m)}\}_{k\in\mathbb{N}}$ which is a subsequence of $\{n_k^{(m-1)}\}_{k\in\mathbb{N}}$ such that $\{x_i^{(n_k^{(m)})}\}_{k\in\mathbb{N}}$ is convergent for all $i = 1, \dots, m$.

Now define $\hat{n}_k := n_k^{(d)}$. Then $\{\hat{n}_k\}_{k\in\mathbb{N}}$ is strictly increasing and $\{x_i^{(\hat{n}_k)}\}_{k\in\mathbb{N}}$ is convergent for each $i = 1, \dots, d$. This means we can define

$$\hat{x}_i := \lim_{k\to\infty} x_i^{(\hat{n}_k)} \tag{17.18}$$

and set $\hat{x} := (\hat{x}_1, \dots, \hat{x}_d)$. We now readily check that

$$\|x^{(\hat{n}_k)} - \hat{x}\|_\infty = \max\bigl\{|x_i^{(\hat{n}_k)} - \hat{x}_i| \colon i = 1, \dots, d\bigr\} \underset{k\to\infty}{\longrightarrow} 0 \tag{17.19}$$

which in light of (17.14) implies

$$\|x^{(\hat{n}_k)} - \hat{x}\| \underset{k\to\infty}{\longrightarrow} 0. \tag{17.20}$$

But the triangle inequality for the ∞-norm shows

$$\|x^{(\hat{n}_k)}\|_\infty - \|x^{(\hat{n}_k)} - \hat{x}\|_\infty \leqslant \|\hat{x}\|_\infty \leqslant \|x^{(\hat{n}_k)}\|_\infty + \|x^{(\hat{n}_k)} - \hat{x}\|_\infty \tag{17.21}$$

which via (17.17) and (17.19) yields $\|\hat{x}\|_\infty = 1$, and a similar argument for $\|\cdot\|$ gives

$$\|x^{(\hat{n}_k)}\| - \|x^{(\hat{n}_k)} - \hat{x}\| \leqslant \|\hat{x}\| \leqslant \|x^{(\hat{n}_k)}\| + \|x^{(\hat{n}_k)} - \hat{x}\| \tag{17.22}$$

implying $\|\hat{x}\| = 0$ by (17.17) and (17.20). This is the desired contradiction because $\|\hat{x}\| = 0$ forces $\hat{x} = 0$ while $\|\hat{x}\|_\infty = 1$ gives $x \neq 0$.

Having proved that $c > 0$ we now note that, since for any $x \neq 0$, the vector $z := \frac{1}{\|x\|_\infty} x$ obeys $\|z\|_\infty = 1$, we have

$$\forall x \in \mathbb{R}^d \smallsetminus \{0\}\colon\;\; \|x\| = \|x\|_\infty \Big\| \frac{1}{\|x\|_\infty} x \Big\| \geqslant c\|x\|_\infty. \tag{17.23}$$

proving the lower bound in (17.13). (For $x = 0$ this bound holds trivially.) □

As part of the previous proof, we have established the following facts:

Corollary 17.12 *All norm-metrics on $\mathbb{R}^d$ have the same Cauchy sequences and the same convergent sequences.*

Proof. That a sequence that is Cauchy (or convergent) in $\|\cdot\|$-metric is Cauchy (or convergent) in $\|\cdot\|_\infty$-metric and *vice versa* follows directly from (17.13). □

Corollary 17.13 *A sequence in $\mathbb{R}^d$ converges in any norm metric if and only if each coordinate thereof converges in the reals endowed with the Euclidean norm.*

Proof. The above proof shows this for the $\|\cdot\|_\infty$-metric; the extension to other norm metric then comes from (17.13). □

This now readily gives:

Proof of Theorem 17.10. If $\{x_n\}_{n\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}}$ is Cauchy in a norm metric ϱ, then it is Cauchy in $\|\cdot\|_\infty$-metric by Corollary 17.12. The argument following (17.19) then shows that the coordinate sequences are Cauchy in the reals endowed with the Euclidean metric. By Theorem 17.2, the coordinates converge and, by Corollaries 17.12 and 17.13, so does $\{x_n\}_{n\in\mathbb{N}}$ in $(\mathbb{R}^d, \varrho)$. □

We also record another fact proved above:

Corollary 17.14 (Bolzano-Weierstrass theorem in $\mathbb{R}^d$) *Every bounded sequence in $\mathbb{R}^d$ endowed with a norm-metric contains a convergent subsequence.*

Proof. This follows from Corollary 17.13 and Theorem 17.7. □

We emphasize that all of the above developments apply solely to norm-metrics; just as for the metric ϱ' on $\mathbb{R}$ not being complete, it is easy to come up with a metric ϱ'' on $\mathbb{R}^d$ that is not complete.

Another remark concerns the proof of Proposition 17.11. A reader might wonder how come that, when the upper bound is proved by essentially algebraic means, the lower bound requires so much analysis. To see that this is necessary, observe that if instead of $\mathbb{R}^2$ we work in the vector space $\mathbb{Q}^2$ over the field $\mathbb{Q}$, then

$$\forall x = (x_1, x_2) \in \mathbb{Q}^2\colon \quad \|x\| := |x_1 + x_2\sqrt{2}| \tag{17.24}$$

actually defines a norm. (This hinges on the fact that there are no rationals $a, b \in \mathbb{Q}$ such that $a + b\sqrt{2} = 0$.) Yet c in (17.15) vanishes and the lower bound in (17.13) fails for this norm because there exists a sequence $\{b_n\}_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ such that $|b_n| < 1$ and $b_n \to 1/\sqrt{2}$ for which $x_n := (1, b_n)$ obeys $\|x_n\| \to 0$ while $\|x_n\|_\infty = 1$. The comparability of the norms is thus tied to the completeness of $\mathbb{R}$ and $\mathbb{R}^d$ for all $d \geqslant 1$.

We also note that while the conclusion of Proposition 17.11 extends to all finite-dimensional vector spaces (as these are isomorphic with $\mathbb{R}^d$ for d being their dimension), the conclusion fails in infinite-dimensional generalizations thereof.