16. SEQUENCES AND POINT-SET TOPOLOGY

The previous section defines a number of concepts having to do with the topology (i.e., study of open and closed sets) in a metric space. We will now link these to the notion based on open balls in the underlying metric, and then also to convergent sequences.

16.1 Point classification.

We being by introducing a classification of points relative to a given set:

Definition 16.1 *Let $A \subseteq X$ and $x \in X$. We say that x is*

- *an adherent point of A if $\forall r > 0\colon B(x,r) \cap A \neq \emptyset$,*
- *a boundary point of A if $\forall r > 0\colon B(x,r) \cap A \neq \emptyset \wedge B(x,r) \cap (X \smallsetminus A) \neq \emptyset$,*
- *an interior point of A if $\exists r > 0\colon B(x,r) \subseteq A$,*
- *an exterior point of A if $\exists r > 0\colon B(x,r) \cap A = \emptyset$.*
- *a limit point of A if $\forall r > 0\colon (B(x,r) \cap A) \smallsetminus \{x\} \neq \emptyset$,*
- *an isolated point of A if $\exists r > 0\colon B(x,r) \cap A = \{x\}$.*

As it turns out, most of these are just different words for notions we already introduced using the notions from topology:

Lemma 16.2 *Let $A \subseteq X$. Then*

(1) $\{x \in X\colon \text{adherent point of } A\} = \overline{A}$,
(2) $\{x \in X\colon \text{boundary point of } A\} = \partial A$,
(3) $\{x \in X\colon \text{interior point of } A\} = \text{int}(A)$,
(4) $\{x \in X\colon \text{exterior point of } A\} = \text{ext}(A) := \text{int}(X \smallsetminus A) = X \smallsetminus \overline{A}$.

Moreover,

$$\overline{A} = \{x \in X\colon \text{limit point of } A\} \cup \{x \in X\colon \text{isolated point of } A\} \tag{16.1}$$

where the two sets in the union are disjoint.

Proof. (1) Let x be an adherent point of A and let C be a closed set with $A \subseteq C$. Then $x \notin X \smallsetminus C$ for otherwise, by the fact that $X \smallsetminus C$ is open, there would be $r > 0$ with $B(x,r) \subseteq X \smallsetminus C$ implying $B(x,r) \cap A \subseteq B(x,r) \cap C = \emptyset$, a contradiction with x being adherent. Taking $C := \overline{A}$ we get $x \in \overline{A}$, proving

$$\{x \in X\colon \text{adherent point of } A\} \subseteq \overline{A}. \tag{16.2}$$

For the other inclusion, denote

$$O := X \smallsetminus \{x \in X\colon \text{adherent point of } A\}. \tag{16.3}$$

Then for every $x \in O$, there is $r > 0$ such that $B(x,r) \cap A = \emptyset$ (otherwise x would be adherent). Since $B(x,r)$ is open, every point therein is separated by an open ball from A and so $B(x,r)$ contains no adherent points of A. This means that $B(x,r) \subseteq O$ and so O is open. Thus

$$\{x \in X\colon \text{adherent point of } A\} \text{ is closed.} \tag{16.4}$$

The fact that $x \in B(x,r)$ for all $r > 0$ shows that all the points in A are automatically adherent, and so we get $A \subseteq \{x \in X\colon \text{adherent point of } A\}$. Since the closure of A is the smallest closed set containing A, this yields

$$\overline{A} \subseteq \{x \in X \colon \text{adherent point of } A\} \tag{16.5}$$

Along with (16.2), this proves (1).

(2) By inspecting the definition of a boundary and adherent point, we readily check that a point is a boundary point if and only if it is adherent to A and to $X \smallsetminus A$. Using (1) we thus get

$$\{x \in X \colon \text{boundary point of } A\} = \overline{A} \cap \overline{X \smallsetminus A} \tag{16.6}$$

The claim now follows from (15.24).

(3-4) The definition of an interior point readily implies that

$$\{x \in X \colon \text{interior point of } A\} = X \smallsetminus \{x \in X \colon \text{adherent point of } X \smallsetminus A\} \tag{16.7}$$

As $\text{int}(A) = X \smallsetminus \overline{X \smallsetminus A}$ by (15.23), the claim (3) follows from (1). The claim (4) is (3) applied to the complement of A.

It remains to prove (16.1). Note that, for all x, we have

$$\begin{aligned} &\forall r > 0 \colon B(x,r) \cap A \neq \emptyset \\ &\quad \Leftrightarrow \big(\exists r > 0 \colon B(x,r) \cap A = \{x\}\big) \\ &\qquad \vee \Big(\big(\forall r > 0 \colon B(x,r) \cap A \neq \emptyset\big) \wedge \neg\big(\exists r > 0 \colon B(x,r) \cap A = \{x\}\big)\Big) \end{aligned} \tag{16.8}$$

Using rules for negation of quantified clauses, the last line can be converted to

$$\forall r > 0 \colon \big(B(x,r) \cap A \neq \emptyset \ \wedge \ B(x,r) \cap A \neq \{x\}\big) \tag{16.9}$$

This is equivalent to $\forall r > 0 \colon (B(x,r) \smallsetminus \{x\}) \cap A \neq \emptyset$, thus proving the claim. The fact that the decomposition in (16.1) is into disjoint sets is checked similarly. □

It is clear from the definition that an isolated point of A always belongs to A. However, the last argument in the previous proofs allows us to characterize isolated and limits points further:

Lemma 16.3 *For all $A \subseteq X$,*

$$\forall x \in A \colon \ \big(\exists r > 0 \colon \ A \cap B(x,r) \text{ non-empty finite}\big) \ \Leftrightarrow \ x \text{ is isolated point of } A \tag{16.10}$$

In particular,

$$\forall x \in X \colon \ x \text{ is a limit point of } A \ \Leftrightarrow \ \forall r > 0 \colon A \cap B(x,r) \text{ is infinite} \tag{16.11}$$

In short, each open ball centered at a limit point of A contains infinitely many points of A.

Proof. Let $x \in A$. If $A \cap B(x,r)$ is finite for some $r > 0$, then there is $n \in \mathbb{N}$ and a bijection $f \colon [0,n) \to A$. Denote $\forall k \in [0,n) \colon x_k := f(k)$ and set $r' := r$ if $n = 0$ and $r' := \min\{\rho(x,x_k) \colon k \in [0,n) \wedge x_k \neq x\}$ if $n > 0$. Then, as is readily checked, $A \cap B(x,r') = \{x\}$ and so x is an isolated point. The converse direction follows immediately from the definition of an isolated point. □

Corollary 16.4 *If $A \subseteq X$ is finite, then all points of A are isolated.*

Proof. Since singletons are closed and finite unions of closed sets are closed, if A is finite then it is closed. In particular, all points of A are adherent. Thanks to finiteness of A, the characterization (16.11) rules out limit points, so by (16.1), all points of A are isolated. (One can also prove this directly by setting $r := \min\{\varrho(x,y) : x, y \in A \wedge x \neq y\}$ and noting that then $\forall x \in A : A \cap B(x,r) = \{x\}$.) $\square$

16.2 Sequential characterization of closedness.

Let us check out a few examples demonstrating the above notions. In all of these we take $X := \mathbb{R}$ with ϱ being the Euclidean metric.

- $A := \{\frac{1}{n+1} : n \in \mathbb{N}\}$. Here each point of A is isolated but $\overline{A} = A \cup \{0\}$ and 0 is a limit point of A. Every point of A lies in ∂A and $\operatorname{int}(A) = \emptyset$.
- $A := \{(-1)^n \frac{n}{n+1} : n \in \mathbb{N}\}$. Here, again, each point of A is isolated and $\overline{A} = A \cup \{+1, -1\}$ and the points $+1$ and -1 are limit points.
- $A := \{\frac{1}{n+1} + \frac{\sqrt{2}}{m+1} : n, m \in \mathbb{N} \wedge n \leqslant m\}$. Here $\{\frac{1}{n+1} : n \in \mathbb{N}\} \cup \{0\}$ are all the limit points while the remaining points of A are isolated.
- $A := \{n\sqrt{2} \bmod 1 : n \in \mathbb{N}\}$. Here $A \subseteq [0,1]$ and $\overline{A} = [0,1]$. Every point of $[0,1]$ is a limit point of A. Still $\operatorname{int}(A) = \emptyset$.

We leave the proofs of the above claims to the reader.

Notice that in the examples we often relied on convergence of sequences from the set. As it turns out, this gives us another way to think of closed sets and closures:

Theorem 16.5 (AC) *Let (X, ϱ) be a metric space. Then for all $A \subseteq X$:*

$$A \text{ is closed} \quad \Leftrightarrow \quad \Big(\forall \{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}\ \forall x \in X : x_n \to x \Rightarrow x \in A\Big) \tag{16.12}$$

In words, a set $A \subseteq X$ is closed if and only if all convergent sequences from A have a limit in A.

Proof. Let us start with the proof of $\Rightarrow$. Suppose $A \subseteq X$ is closed and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence from A such that $x_n \to x$. If $x \notin A$ then x lies in $X \smallsetminus A$ which is open and so there is $r > 0$ such that $B(x,r) \cap A = \emptyset$. But $x_n \to x$ means that $x_n \in B(x,r)$ when n is sufficiently large in contradiction with $x_n \in A$. Summarizing, $x \notin A$ implies $\neg(x_n \to x)$ which is equivalent to $x_n \to x \Rightarrow x \in A$, proving $\Rightarrow$ in (16.12).

Let us now consider the implication $\Leftarrow$ which we will again prove by proving the contrapositive. Suppose A is NOT closed. Then $X \smallsetminus A$ is NOT open and so

$$\exists x \in X \smallsetminus A\ \forall r > 0 : B(x,r) \cap A \neq \emptyset. \tag{16.13}$$

This means that there exists $x \notin A$ that is adherent to A. Specializing (16.13) to r in the set $\{2^{-n} : n \in \mathbb{N}\}$, the Axiom of Choice yields

$$\bigtimes_{n\in\mathbb{N}} A \cap \big(B(x, 2^{-n}) \smallsetminus \{x\}\big) \neq \emptyset \tag{16.14}$$

meaning that there exists $f : \mathbb{N} \to A$ with $\forall n \in \mathbb{N} : f(n) \in B(x, 2^{-n}) \smallsetminus \{x\}$. Setting $x_n := f(n)$, we have $\rho(x, x_n) < 2^{-n}$ and so $x_n \to x$. Summarizing, assuming that A is NOT closed we showed that there exists an A-valued sequence $\{x_n\}_{n\in\mathbb{N}}$ and $x \in X$ such that $x_n \to x \wedge \neg(x \in A)$. This proves $\Leftarrow$ in (16.12) as desired. $\square$

We used "AC" in the label of the theorem to mark that the proof required the Axiom of Choice. This is necessary when no additional structure is assumed about (X, ϱ) and A. However, in most spaces that we encounter in practice (e.g., when X is separable, see Definition 16.7 below) the choice of x_n can be performed constructively and then the Axiom of Choice is no longer required.

Corollary 16.6 (AC)(Density of a set in its closure) *Let $A \subseteq X$. Then*

$$\forall x \in X\colon \quad x \in \overline{A} \Leftrightarrow \exists \{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}\colon\ x_n \to x. \tag{16.15}$$

Proof. For $\Rightarrow$ use that, by Lemma 16.2, $\overline{A}$ is the set of adherent points and apply the argument after (16.13). For $\Leftarrow$ note that if $x_n \to x$ for some $\{x_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$, then x is an adherent point of A and so $x \in \overline{A}$, again by Lemma 16.2. □

The reason why we used the word "density" to label the property in Corollary 16.6 arises from:

Definition 16.7 *We say that a set $B \subseteq X$ is* dense *in $A \subseteq X$ if $A \subseteq \overline{B}$.*

Thus we also get:

Corollary 16.8 (AC) *Let $A, B \subseteq X$. Then B is dense in A if and only if for each $x \in A$ there exists a B-valued sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to x$.*

Typically, we will apply this to $A = X$ or $B \subseteq A$ (or both). A standard example of a dense subset of $\mathbb{R}$ (with the Euclidean metric) is $\mathbb{Q}$, although $\mathbb{R} \smallsetminus \mathbb{Q}$ does as well. However, the former set demonstrates an important property of the reals:

Definition 16.9 (Separability) *We say that a metric space (X, ϱ) is* separable *if it contains a countable dense set, i.e., if*

$$\exists A \subseteq X\colon A \text{ countable} \wedge \overline{A} = X. \tag{16.16}$$

(This is one example where we do allow a finite A to be regarded as countable.)

The reals are thus separable. A homework exercise asks to show that the same applies to d-dimensional Euclidean spaces $\mathbb{R}^d$ under any norm metric. This extends even to some, but not all, infinite-dimensional generalizations thereof. (Note that $\mathbb{R}$ endowed with the discrete metric is definitely *not* separable.)

16.3 Relative notions.

Given a metric space (X, ϱ), associated with each (non-empty) $Y \subseteq X$ is a natural metric space (Y, ϱ_Y) where ϱ_Y is simply the restriction of ϱ to pairs of points from Y. We call ϱ_y the *induced metric*. Now define the following *relative* topological concepts:

Definition 16.10 (Relative open/closed set) *Let $Y \subseteq X$ be as above. We say that $A \subseteq Y$ is* relatively open *if A is open in (Y, ϱ_Y), and* relatively closed *if A is closed in (Y, ϱ_Y).*

In order to give an example, consider the following setting: $X := \mathbb{R}$ endowed with the standard Euclidean metric, $Y := \mathbb{Q}$. Then $A := [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$ is relatively closed and relatively open in $\mathbb{Q}$ while it is neither open nor closed in $\mathbb{R}$. Same applies to the set $A := (\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$ as well as to $A := (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$. This (of course) has to do with the fact that $\sqrt{2}, \sqrt{3} \notin \mathbb{Q}$; indeed, the set $A := [0,1) \cap \mathbb{Q}$ is neither relatively open nor relatively closed while $A := [0, \sqrt{2}) \cap \mathbb{Q}$ is relatively closed but not relatively open.

Another example to consider is $Y := [0,2)$. Then $A := [0,1)$ is relatively open and $A := [1,2)$ is relatively closed. These facts can be verified directly or by invoking the following general characterization of relatively open and closed sets:

Theorem 16.11 *Let (X, ϱ) be a metric space and let $Y \subseteq X$ be nonempty. Then*

$$\forall A \subseteq Y\colon \quad A \text{ relatively open} \;\Leftrightarrow\; \Bigl(\exists O \subseteq X\colon\; O \text{ open} \;\wedge\; A = O \cap Y\Bigr) \tag{16.17}$$

Similarly, we get

$$\forall A \subseteq Y\colon \quad A \text{ relatively closed} \;\Leftrightarrow\; \Bigl(\exists C \subseteq X\colon\; C \text{ closed} \;\wedge\; A = C \cap Y\Bigr) \tag{16.18}$$

In short, a set is relatively open/closed if and only if it is a restriction of an open/closed set.

Proof. For the purpose of this proof, let $B_Y(x,r) := \{y \in Y\colon \varrho(x,y) < r\}$ denote the open ball (Y, ϱ_Y) and let $B_X(x,r) := \{y \in X\colon \varrho(x,y) < r\}$ be the open ball in (X, ϱ). Note that

$$\forall x \in Y\ \forall r > 0\colon\; B_Y(x,r) = Y \cap B_X(x,r) \tag{16.19}$$

as is directly checked from the definition.

Let us start with "$\Rightarrow$" in (16.17). If $A \subseteq Y$ is a relatively open set, then it is open in (Y, ϱ_Y) which means that

$$\forall x \in A\ \exists r_x > 0\colon\; B_Y(x, r_x) \subseteq A. \tag{16.20}$$

(This r_x can be picked constructively; e.g., as $r_x := \frac{1}{2}\sup\{r \in (0,1]\colon B(x,r) \subseteq A\}$.) Set

$$O := \bigcup_{x \in A} B_X(x, r_x). \tag{16.21}$$

Since $B_X(x, r_x)$ is open in (X, ϱ) (see Lemma 15.4) and the union of a family of open sets is open, we have

$$O \text{ is open in } (X, \varrho). \tag{16.22}$$

Since $x \in B_X(x, r_x)$, we have $A \subseteq O$ and so

$$A \subseteq O \cap Y. \tag{16.23}$$

For the opposite inclusion note

$$O \cap Y = \bigcup_{x \in A} B_X(x, r_x) \cap Y = \bigcup_{x \in A} B_Y(x, r_x) \subseteq A \tag{16.24}$$

where we used (16.1) and then (16.20) at the very end. Combining (16.23) and (16.24) we get "$\Rightarrow$" in (16.17).

In order to prove "$\Leftarrow$" in (16.17), let $O \subseteq X$ be open with $A = O \cap Y$. Then for each $x \in A$ we have $x \in O$ and so there is $r > 0$ such that $B_X(x,r) \subseteq O$. But then (16.19) shows $B_Y(x,r) = B_X(x,r) \cap Y \subseteq O \cap Y = A$ proving that A is relatiely open.

The proof of (16.18) is handled by complementation. Indeed,

$$A \text{ relatively closed} \;\Leftrightarrow\; Y \smallsetminus A \text{ relatively open}. \tag{16.25}$$

By (5.1) the statement on the right-hand side is equivalent to the existence of $O \subseteq X$ open in (X, ϱ) such that $Y \smallsetminus A = O \cap Y$. But that is in turn equivalent to

$$A = Y \smallsetminus (O \cap Y) = Y \cap (X \smallsetminus O) \tag{16.26}$$

which is the right-hand side of (16.18) because $X \smallsetminus O$ is closed in (X, ϱ). □

We finish by noting that in topology, the relative notions are *defined* by the right-hand sides of (5.1–16.18). The relative topology on Y is then the projection of the topology on X by way of intersecting all sets by Y.