15. Basic topology

Having discussed the notion of Cauchy and convergent sequences, we now turn to the following natural questions:

- In what metric spaces or subsets thereof do all Cauchy sequences have a limit?
- What sets in a given metric space contain the limit of all convergent sequences contained therein.
- In what sets or spaces do sequences admit convergent subsequences.

These questions will ultimately be answered by the words *complete*, *closed* and *compact*, respectively. Here we develop the necessarily tools starting with metric spaces and then move to their generalizations using notions from *topology*.

15.1 Open balls and open sets.

We start with the basic definition:

Definition 15.1 (Open ball) *Let (X, ϱ) be a metric space. Given an $x \in X$ and a real number $r > 0$, the* open ball $B(x, r)$ *of radius $r > 0$ centered at $x \in X$ is the set*

$$B(x,r) := \{y \in X : \varrho(x,y) < r\}. \tag{15.1}$$

Note that we have $x \in B(x, r)$ for all $r > 0$. We do not consider open balls for radii $r \leqslant 0$ as these are empty and thus not very interesting.

Before we start using the notion of open ball, it is instructive to check what the open balls look like in some of the basic examples of the metric spaces:

- *discrete metric*: As the metric takes only values 0 and 1, here we get

$$B(x,r) = \begin{cases} \{x\}, & \text{if } 0 < r \leqslant 1, \\ X, & \text{if } r > 1. \end{cases} \tag{15.2}$$

  In short, the ball is either a single point (namely, the center) or the whole space.
- *The real line*: Using the Euclidean metric $\varrho(x, y) := |x - y|$, we have

$$B(x,r) = (x-r, x+r) \tag{15.3}$$

  so Euclidean balls in $\mathbb{R}$ are simply open intervals. The same is true for the metric ϱ' from (14.14) although there the interval is no longer centered at x and no longer of (Euclidean) length $2r$ (verify this precisely!).
- *Euclidean space*: Consider the normed space $(\mathbb{R}^d, \|\cdot\|_p)$ for $p \in [1, \infty]$ and denote by $B_p(x, r)$ the open ball in $\mathbb{R}^d$ with respect to the norm-metric ϱ_p derived from $\|\cdot\|_p$; see (14.18) and (14.20). For $p = 2$, $B_2(x, r)$ is the usual, perfectly round, Euclidean ball. However, when $p = \infty$, we have

$$B_\infty(x,r) = \bigtimes_{i=1}^{d} (x_i - r, x_i + r) \tag{15.4}$$

  meaning that the ball in the ϱ_∞-metric is an open cube centered at x. (In this sense the ∞-metric is a more natural extension of (14.13) because, just as $\mathbb{R}^d$ is the Cartesian product of $\mathbb{R}$'s, the ∞-ball in $\mathbb{R}^d$ is the Cartesian product of (15.3).

For $p = 1$ the ball $B_1(x,r)$ takes a shape of a diamond centered at x. As p increases above 1, the corners of the diamond become rounded to become the Euclidean ball at $p = 2$. As p increases further, the p-ball gradually morphs to a cube.

With the notion of the open ball in hand, we now put forward:

Definition 15.2 (Open and closed sets) *Let (X, ϱ) be a metric space. A set $A \subseteq X$ is said to be* open *if*

$$\forall x \in A\, \exists r > 0 \colon B(x,r) \subseteq A, \tag{15.5}$$

i.e., if along with every point the set contains an open ball centered at that point. A set $A \subseteq X$ is said to be closed *if $X \smallsetminus A$ is open.*

We remark that that latter already introduces closed sets using the method typical for topology. The text book uses a definition based on the concept of a *limit point* which we will show to be equivalent in the next section.

Here are some basic examples:

Lemma 15.3 *Every singleton is closed, i.e., $\forall x \in X \colon \{x\}$ is closed.*

Proof. This is equivalent to saying that, $\forall x \in X \colon X \smallsetminus \{x\}$ is open. To prove that, let $y \in X \smallsetminus \{x\}$. Since $y \neq x$, we have $\varrho(x,y) > 0$ so if we let $r := \varrho(x,y)$, then $r > 0$. But then $\varrho(x,y) = r$ and so $x \notin B(y,r)$ meaning that $B(y,r) \subseteq X \smallsetminus \{x\}$. As this is true for every $y \in X \smallsetminus \{x\}$, the set $X \smallsetminus \{x\}$ is open and its complement $\{x\}$ is closed. □

Lemma 15.4 *Every open ball is open, i.e., $\forall x \in X\, \forall r > 0 \colon B(x,r)$ is open.*

Proof. This is small variation on the previous proof. Let $y \in B(x,r)$. Then $\varrho(x,y) < r$ and so $\epsilon := r - \varrho(x,y) > 0$. Let $z \in B(y,\epsilon)$. The triangle inequality then implies

$$\begin{aligned} \varrho(x,z) &\leqslant \varrho(x,y) + \varrho(y,z) \\ &< \varrho(x,y) + \epsilon = \varrho(x,y) + [r - \varrho(x,y)] = r \end{aligned} \tag{15.6}$$

and so $z \in B(x,r)$. It follows that $B(y,\epsilon) \subseteq B(x,r)$ and so $B(x,r)$ is open. □

Lemma 15.5 *Let ϱ be a discrete metric on X. Then every subset of X is open and closed.*

Proof. Let $A \subseteq X$ be arbitrary. Then for all $x \in A$ we have $B(x,1/2) = \{x\} \subseteq A$ by (15.2) and so A is open. As this holds for all $A \subseteq X$, we get that $X \smallsetminus A$ is open and so A is also closed, as claimed. □

The example of the discrete metric may be misleading it that it might make the reader believe that most (or even all) sets are either open or closed. But this is far from the truth in general; indeed, being open or closed is a very special property and most sets in general metric space are neither open nor closed. For instance, the interval $(0,1]$ is neither open nor closed in $\mathbb{R}$ with respect to the usual metric. So, please beware that

$$A \text{ is NOT open} \quad \not\Rightarrow \quad A \text{ is closed} \tag{15.7}$$

and

$$A \text{ is open} \quad \not\Rightarrow \quad A \text{ is NOT closed} \tag{15.8}$$

In other words, the notions of open and closed sets are neither exhaustive (as other sets than these may exist) nor exclusive (as there could be sets that are both open and closed).

Lemma 15.4 naturally guides us to:

Definition 15.6 *Let (X, ϱ) be a metric space. Given $x \in X$ and $r \in \mathbb{R}$ with $r \geqslant 0$, the set*

$$\{y \in X \colon \varrho(y, x) \leqslant r\} \tag{15.9}$$

is called the closed ball *of radius r centered at x.*

As shown in the homework exercise, a closed ball is indeed closed, justifying its name.

15.2 Topology.

We will now characterize the open sets in a metric space as follows:

Lemma 15.7 *Let (X, ϱ) be a metric space and set $\mathcal{T} := \{O \subseteq X \colon \text{open}\}$. Then*

(1) $\varnothing, X \in \mathcal{T}$
(2) $\forall A \subseteq \mathcal{T} \colon \bigcup A \in \mathcal{T}$
(3) $\forall A \subseteq \mathcal{T} \colon A \text{ finite } \Rightarrow \bigcap A \in \mathcal{T}$

In words, the set of open sets in a metric space (X, ϱ) contains $\varnothing$ and X and is closed under arbitrary unions and finite intersections.

Proof. (1) Since $\varnothing$ contains no points, it is trivially open (there is no x for which it would have to contain an open ball centered at x). Similarly, X is open as it by definition contains all open balls.

(2) Let A be a collection of open sets and let $x \in \bigcup A$. Then there is $O \in A$ such that $x \in O$. But O is open and so there is an $r > 0$ such that $B(x, r) \subseteq O$. It follows that

$$B(x, r) \subseteq O \subseteq \bigcup A \tag{15.10}$$

and so $\bigcup A$ is open.

(3) Let $A \subseteq \mathcal{T}$ be finite. This means that there is $n \in \mathbb{N}$ and a map that assigns each natural $k = 0, \dots, n$ to a set $O_k \in A$ such that $A = \{O_k \colon k = 0, \dots, n\}$. If $x \in \bigcap A$ then for each $k = 0, \dots, n$ we have $x \in O_k$ and, since O_k is open, there is $r_k > 0$ such that $B(x, r_k) \subseteq O_k$. Now set

$$r := \min_{k \leqslant n} r_k \tag{15.11}$$

and note that, being the minimum of a finite number of positive reals, $r > 0$. As $r \leqslant r_k$, this shows

$$\forall k = 0, \dots, n \colon B(x, r) \subseteq B(x, r_k) \subseteq O_k. \tag{15.12}$$

But then $B(x, r) \subseteq \bigcap_{k=0}^{n} O_k = \bigcap A$ and so $\bigcap A$ is open. □

The properties of open sets in the previous lemma can be abstracted as:

Definition 15.8 (Topology) *Let X be a set. A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ is said to be a* topology *on X if $\varnothing, X \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary unions and finite intersections.*

When a topology $\mathcal{T}$ is given, we refer to sets in $\mathcal{T}$ as *open* and, in accord with our earlier definition, call complements of open sets *closed*. The closed sets then obey:

Lemma 15.9 *The set of closed sets corresponding to a topology on X contains $\varnothing$ and X and is closed under arbitrary intersections and finite unions.*

Proof. With the help of de Morgan law

$$\forall \mathcal{A} \subseteq \mathcal{P}(X): \quad X \smallsetminus \bigcap \mathcal{A} = \bigcup \{X \smallsetminus C : C \in \mathcal{A}\} \tag{15.13}$$

this follows directly from the corresponding properties of open sets. □

Every non-empty set X supports two topologies: first, the *coarsest* topology $\{\varnothing, X\}$ and the *finest* or *discrete* topology $\mathcal{P}(X)$. The name of the latter arises from the fact that this topology comes from a metric — namely, the discrete metric, thanks to Lemma 15.5 — and is thus *metrizable*. The coarsest topology does not come from a metric unless X is a singleton. (This is because all singletons are closed in every metric space.)

We now introduce the following concepts:

Definition 15.10 (Interior and closure) *Let $A \subseteq X$. The* interior of A *is then the set*

$$\text{int}(A) := \bigcup \{O \subseteq X : \text{ open } \wedge \ O \subseteq A\}, \tag{15.14}$$

namely, the union of all open sets contained in A. The closure of A *is then the set*

$$\overline{A} := \bigcap \{C \subseteq X : \text{ closed } \wedge \ A \subseteq C\}, \tag{15.15}$$

namely, the intersections of all closed sets containing A.

Note that each $A \subseteq X$ contains at least one open set (namely, $\varnothing$) and is contained in at least one closed set (namely, X). Lemmas 15.5-15.9 then readily show

$$\forall A \subseteq X: \ \text{int}(A) \text{ is open } \wedge \ \overline{A} \text{ is closed.} \tag{15.16}$$

We remark that other notations may be encountered in the literature for the interior (e.g., A°) and the closure (e.g., cl(A)). In addition, we have the following facts:

Lemma 15.11 *For each $A \subseteq X$,*

(1) $\text{int}(A) \subseteq A \subseteq \overline{A}$,
(2) *A is open* $\Leftrightarrow A = \text{int}(A)$,
(3) *A is closed* $\Leftrightarrow A = \overline{A}$.

Proof. (1) is the consequence of (15.14) and (15.15). In light of (15.16) we only have to prove ⇒ in (2-3). But this again follows from (15.14) and (15.15): if A is open, then A is part of the collection in (15.14) and so $A \subseteq \text{int}(A)$. By (1) we get $A = \text{int}(A)$, proving ⇒ in (2). If A is in turn closed, then A belongs to the collection in (15.15) and so $\overline{A} \subseteq A$. By (1) we then get $A = \overline{A}$, proving ⇒ in (3). □

The interior of A is thus the largest open set contained in A while the closure of A is the smallest closed set containing A. Specializing to balls in a metric space (X, ϱ), for all $x \in X$ and $r \geqslant 0$ we thus have

$$\overline{B(x,r)} \subseteq \{y \in X : \varrho(x,y) \leqslant r\} \tag{15.17}$$

meaning that the closure of the open ball is contained in the closed ball. The inequality is strict when $r = 0$ in general but there are metrics in which the inclusion can be strict even for $r > 0$!

Part (1) of Lemma 15.11 leads to another very natural concept:

Definition 15.12 (Topological boundary) *For each $A \subseteq X$, the set*

$$\partial A := \overline{A} \smallsetminus \operatorname{int}(A) \tag{15.18}$$

is the (topological) boundary of A.

Note that writing

$$\overline{A} \smallsetminus \operatorname{int}(A) = \overline{A} \cap (X \smallsetminus \operatorname{int}(A)) \tag{15.19}$$

shows that

$$\forall A \subseteq X\colon\ \partial A \text{ is closed.} \tag{15.20}$$

Also note that while A may not be disjoint from ∂A, we always have

$$\operatorname{int}(A) \cap \partial A = \varnothing. \tag{15.21}$$

An intuitive image of the boundary of A is the "curve or surface enclosing A" but this is true only for nice subsets of the Euclidean space. For instance, the boundary of the set can be the set itself (e.g., for $\mathbb{N}$ regarded as a subset of $\mathbb{R}$ with the Euclidean metric we have $\partial\mathbb{N} = \mathbb{N}$) or even much larger than that (e.g., for $\mathbb{Q} \subseteq \mathbb{R}$ where $\partial\mathbb{Q} = \mathbb{R}$). The boundary can also be empty, e.g., for the finest topology (which, as argued above, corresponds to the discrete metric of X) we have $\partial A = 0$ for each $A \subseteq X$.

Some additional properties of interior and closure are stated in:

Lemma 15.13 *For each $A, B \subseteq X$:*

$$A \subseteq B \quad\Rightarrow\quad \operatorname{int}(A) \subseteq \operatorname{int}(B) \quad\wedge\quad \overline{A} \subseteq \overline{B}. \tag{15.22}$$

Moreover, for each $A \subseteq X$,

$$X \smallsetminus \operatorname{int}(A) = \overline{X \smallsetminus A} \quad\wedge\quad X \smallsetminus \overline{A} = \operatorname{int}(X \smallsetminus A). \tag{15.23}$$

In particular, we have

$$\partial A = \overline{A} \cap \overline{X \smallsetminus A} = \partial(X \smallsetminus A). \tag{15.24}$$

We leave the proof of this lemma to an exercise. There is (quite naturally) no relation between the boundaries ∂A and ∂B whether A is a subset of B or not.

Describing the whole class of open sets in a given metric space is usually hard if not impossible. One example where it can be done is the real line with the usual metric:

Theorem 15.14 *Consider the metric space $(\mathbb{R}, d)$ where $\varrho(x,y) := |x-y|$. Then*

$$\forall A \subseteq \mathbb{R}\colon \qquad A \text{ open} \quad\Leftrightarrow\quad \begin{cases} \exists\{I_n\colon n \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{R})\colon \\ \forall n \in \mathbb{N}\colon I_n = \varnothing \vee I_n = \text{ open interval} \\ \forall m,n \in \mathbb{N}\colon\ I_n \cap I_m \neq \varnothing \ \Rightarrow\ m = n \\ A = \bigcup_{n\in\mathbb{N}} I_n \end{cases} \tag{15.25}$$

In words, every open set in $\mathbb{R}$ is a finite or countable union of disjoint open intervals.

We leave the proof of this theorem to homework. The statement is very special to the one-dimensional Euclidean space. The open sets in $\mathbb{R}^d$ for $d \geqslant 2$ are much harder to characterize.

It is easy to check that any subset of $\mathbb{R}$ that is open for the Euclidean metric is open for the metric (14.14), and *vice versa*. This is not too surprising in itself until we realize that, as shown in a homework exercise, these two metrics have *different* sets of Cauchy sequences. It follows that, while the topological point of view of metric spaces is useful in many ways, it is not good for studying the relation between Cauchy and convergent sequences. In particular, the notion of completeness to be introduced later is tied to the metric structure rather than topology.