12. Cardinality and countability

With the reals in place, we are in a position to start discussing actual real analysis. However, there is still one additional foundational topic that we will need repeatedly in what follows and that is the notion of the "size" of a set. In set theory, this goes under the name *cardinality*.

12.1 Finite and infinite sets.

One way to interpret the loose notion of the "size" of a set is as the number of its elements. In daily life this is something that would be decided by counting which amounts to labeling of the elements of the set by naturals. To make this mathematically precise, we need to introduce the concept of the set of "first n naturals"

$$[0,n) := \{k \in \mathbb{N} \colon k < n\} \tag{12.1}$$

where $k < n := k \leqslant n \wedge k \neq n$ for $\leqslant$ defined as $m \leqslant n := \exists k \in \mathbb{N} \colon n = m + k$. With this, we first divide all sets into two basic categories:

Definition 12.1 *A set A is said to be*

- finite *if there exist $n \in \mathbb{N}$ and a bijection $f \colon [0,n) \to A$, and*
- infinite *if it is not finite.*

Since $[0,0) = \varnothing$, a bijective map $f \colon [0,0) \to \varnothing$ exists trivially and $\varnothing$ is thus finite. Concerning non-empty finite sets, recall a lemma from homework:

Lemma 12.2 *Let $m, n \in \mathbb{N}$ and let $f \colon [0,n) \to [0,m)$ be a function. Then*

(1) f injective $\Rightarrow\ n \leqslant m$
(2) f surjective $\Rightarrow\ m \leqslant n$
(3) f bijective $\Rightarrow\ m = n$

Noting that the inverse of a bijection as well as the composition of two bijections are bijections, part (3) shows that for each set A there is at most one $n \in \mathbb{N}$ for which a bijection $f \colon [0,n) \to A$ exists. It is thus meaningful to give:

Definition 12.3 *Let A be a finite set. The unique $n \in \mathbb{N}$ for which there is a bijection $f \colon [0,n) \to A$ is called the* cardinality *of A with notation $|A|$ (or sometimes $\#A$).*

We leave to the reader to verify that the concept of finite set is closed under finite unions and subset relation:

Lemma 12.4 *For any sets A and B, we have:*

(1) A finite $\wedge$ $B \subseteq A\ \Rightarrow\ B$ finite
(2) A finite $\wedge$ B finite $\Leftrightarrow A \cup B$ finite
(3) A finite $\wedge$ B finite $\Leftrightarrow A \times B$ finite

These also translate into inequalities for cardinality:

Lemma 12.5 *For any finite sets A and B:*

$$B \subseteq A\ \Rightarrow\ |B| \leqslant |A| \tag{12.2}$$

and

$$|A \cup B| \leqslant |A| + |B| \tag{12.3}$$

and

$$|A \times B| = |A| \cdot |B| \tag{12.4}$$

Thus, intuitively, union translates into addition and Cartesian product into multiplication. The inequality in (12.3) arises from possible overcounting. Indeed, equality holds when (and only when) A and B are disjoint.

As to the second eventuality in Definition 12.1, we note that earlier (specifically, in Definition 3.12) we put forward the notion of a set A being *Dedekind infinite* to denote the situation that there exists an injection $f\colon A \to A$ with $\text{Ran}(f) \neq A$. This builds on *Galileo's paradox* that points out an apparent contradiction between the fact that only some naturals are squares and so there seems to be more naturals than squares and the existence of the map $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(n) := n^2$ that puts all squares with all naturals in one-to-one correspondence.

The notion of being Dedekind infinite is attractive for its intrinsic nature; indeed, it does not require anything else than the set itself. Since the existence of an injection $f\colon A \to A$ with $\text{Ran}(f) \neq A$ rules out (with the help of part (1) of Lemma 12.2) that A is finite, we have

$$A \text{ Dedekind infinite } \Rightarrow \ A \text{ infinite} \tag{12.5}$$

The converse seems intuitive as well; indeed, keep listing elements of A to produce a sequence $\{x_n\colon n \in \mathbb{N}\}$ these elements. Then define $f(x_n) := x_{n+1}$ for $n \in \mathbb{N}$ and $f(x) := x$ for x not a member of the sequence $\{x_n\colon n \in \mathbb{N}\}$ to get an injection of A into itself which misses a point — namely, the point x_0.

Unfortunately, this is where a precise argument runs into a problem: the need to repeatedly choose an element from a set cannot be formalized without the use of Axiom of Choice. This is not a mere technicality; indeed, there are models of set theory with Zermelo's axioms but without Axiom of Choice in which the converse to (12.5) fails. For this reason we abandon all use of the notion of Dedekind-infinite sets and stick henceforth with the notions put foward in Definition 12.1.

12.2 Countable sets.

Having elucidated how to interpret the "size" of finite sets, let us move to refining the notion of infinite sets. In analogy with Definition 12.1, here we put:

Definition 12.6 *An infinite set A is said to be*

- countable *if there exists a bijection $f\colon \mathbb{N} \to A$, and*
- uncountable *if it is not countable.*

Informally, a set is countable if it can be enumerated into a sequence. While this notion is defined for infinite sets above, it is commonly extended to include finite sets as well. This is not true about the book, which talks about being *at most countable* in this case. In other texts, the term *denumerable* is used to denote "infinite and countable."

We will focus on countable sets first and derive some basic facts about them. Our first observation is that, just as for finite sets, countability is inherited by the subsets:

Lemma 12.7 *Let A be a countable set. Then*

$$\forall B \subseteq A\colon \quad B \text{ infinite } \Rightarrow\ B \text{ countable} \tag{12.6}$$

In particular, every subset of a countable set is either finite or countable.

Proof. Since A is in a bijective correspondence with $\mathbb{N}$, it suffices to prove this for $A := \mathbb{N}$ and $B \subseteq \mathbb{N}$. Informally, we will use the ordering of $\mathbb{N}$ to enumerate all elements of B into a sequence. The formal construction requires some work.

Define, recursively,

$$B_0 := B \ \wedge\ \forall k \in \mathbb{N}\colon B_{k+1} := \begin{cases} B_k \smallsetminus \{\inf(B_k)\} & \text{if } B_k \neq \emptyset, \\ \emptyset & \text{if } B_k = \emptyset, \end{cases} \tag{12.7}$$

where we recalled that, as shown in Lemma 9.7, each non-empty set of naturals has an infimum which is then contained therein. Then prove

$$B \text{ infinite } \Rightarrow\ \forall k \in \mathbb{N}\colon B_k \text{ infinite} \tag{12.8}$$

by induction: The case $k = 0$ is checked from $B_0 = B$. For the induction step assuming that B_k is infinite for some $k \in \mathbb{N}$. Then $B_k \neq \emptyset$ and so $B_k = B_{k+1} \cup \{\inf(B_k)\}$. If B_{k+1} were finite, then so would B_k by Lemma 12.4, and so B_{k+1} is infinite as well.

We then also similarly claim

$$\forall k \in \mathbb{N}\colon \quad \inf(B_k) + 1 \leqslant \inf(B_{k+1}) \ \wedge\ k \leqslant \inf(B_k) \tag{12.9}$$

The first part is proved directly from $\inf(B_k) \notin B_{k+1}$. The second part is proved by induction; indeed, $0 \leqslant \inf(B_0)$ trivially and, if $k \leqslant \inf(B_k)$ then the first part gives $k + 1 \leqslant \inf(B_k) + 1 \leqslant \inf(B_{k+1})$, proving the induction step.

With the above in hand, we define

$$\forall k \in \mathbb{N}\colon \quad f(k) := \inf(B_k) \tag{12.10}$$

This is defined for all $k \in \mathbb{N}$ by (12.8). Also f is injective because the first part of (12.9) gives $\forall k, \ell \in \mathbb{N}\colon k < \ell \Rightarrow \inf(B_k) < \inf(B_\ell)$. To see that f is surjective, let $n \in B$ and set

$$k(n) := \inf\{k \in \mathbb{N}\colon n \notin B_k\}. \tag{12.11}$$

where the set on the right is non-empty (and the infimum is thus well defined) because $n \notin B_{n+1}$ by the fact that $\inf(B_{n+1}) \geqslant n + 1$ by (12.9). Now observe that $k(n) > 0$ because $n \in B_0 = B$ and so $k(n) - 1 \in \mathbb{N}$. In addition, $n \notin B_{k(n)}$ bu $n \in B_{k(n)-1}$ for otherwise $k(n)$ is not the infimum. Hence

$$n \in B_{k(n)-1} \smallsetminus B_{k(n)} = \big\{\inf(B_{k(n)-1})\big\} \tag{12.12}$$

proving that $n = \inf(B_{k(n)-1}) = f(k(n) - 1)$. Hence $n \in \mathrm{Ran}(f)$ and f is surjective. □

The statement and proof can be summed up as follows. First put forward a very useful notion:

Definition 12.8 (Sequence) *Let A be a set. An A-valued* sequence $\{x_n\}_{n\in\mathbb{N}}$ *(or a sequence taking values in A) is a function $f\colon \mathbb{N} \to A$ with $\mathrm{Dom}(f) = \mathbb{N}$ and such that $\forall n \in \mathbb{N}\colon x_n = f(n)$. We reserve the notation $\{x_n\colon n \in \mathbb{N}\}$ for $\mathrm{Ran}(f)$.*

The above proof then shows:

Corollary 12.9 *Let A be an infinite countable set. For each infinite $B \subseteq A$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $B = \{x_n : n \in \mathbb{N}\}$. (We say that B is* exhausted by *$\{x_n\}_{n\in\mathbb{N}}$.)*

Another observation that we will find useful is as follows:

Corollary 12.10 *For any set B:*

$$B \text{ finite or countable } \Leftrightarrow \exists f\colon B \to \mathbb{N}\colon \textit{ injection}. \tag{12.13}$$

Proof. The direction $\Rightarrow$ follows directly from Definitions 12.1 and 12.6. For $\Leftarrow$ we just need to address the case of B infinite. Here the injection f puts B into bijective correspondence with $f(B) = \mathrm{Ran}(f)$ which, being a subset of $\mathbb{N}$, is countable by Lemma 12.7. Thus B is countable as well. □

We will now proceed to note that, similarly as for finite sets, countability is preserved by certain natural operations on sets. We begin by the Cartesian product:

Lemma 12.11 *Let A and B be countable sets. Then so is $A \times B$.*

Proof. Let $f\colon A \to \mathbb{N}$ and $g\colon A \to \mathbb{N}$ be injections ensured by A and B being countable. Then $h(a,b) := (f(a), g(b))$ defines an injection $h\colon A \times B \to \mathbb{N} \times \mathbb{N}$. (We leave checking that h is an injection to the reader.) It thus suffice so show that there is an injection $\phi\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. This is provided by

$$\phi(m,n) := \frac{1}{2}(m+n)(m+n+1) + m, \tag{12.14}$$

where the first term on the right is a natural by the fact that the product of two consecutive naturals is even. Indeed, this map labels the elements of $\mathbb{N} \times \mathbb{N}$ consecutively according to the following ordering on $\mathbb{N} \times \mathbb{N}$:

$$(m,n) \leqslant (\widetilde{m}, \tilde{n}) \ := \ m+n < \widetilde{m} + \tilde{n} \ \vee \ \Big(m+n = \widetilde{m} + \tilde{n} \ \wedge m \leqslant \widetilde{m}\Big) \tag{12.15}$$

We leave it to the reader to check that this is a partial (in fact, total) order.

To see that ϕ is an injection, it suffices to show that it is strictly monotone. Indeed, if $k := m+n < \tilde{k} := \widetilde{m} + \tilde{n}$, then $k+1 \leqslant \tilde{k}$ and so

$$\begin{aligned} \phi(m,n) &= \frac{1}{2}k(k+1) + m \leqslant \frac{1}{2}k(k+1) + k = \frac{1}{2}k(k+3) \\ &\leqslant \frac{1}{2}(\tilde{k}-1)(\tilde{k}+2) = \frac{1}{2}\tilde{k}(\tilde{k}+1) - 1 < \frac{1}{2}\tilde{k}(\tilde{k}+1) + \widetilde{m} = \phi(\widetilde{m}, \tilde{n}) \end{aligned} \tag{12.16}$$

If $m+n = \widetilde{m} + \tilde{n}$ and $m < \widetilde{m}$, then $\phi(m,n) < \phi(\widetilde{m}, \tilde{n})$ directly from the definition. □

Corollary 12.12 *The set $\mathbb{Q}$ of all rationals is infinite and countable.*

Proof. Since $\mathbb{N}$ is infinite by Lemma 12.2, $\mathbb{N} \subseteq \mathbb{Q}$ and Lemma 12.4 force that $\mathbb{Q}$ is infinite as well. For the second part we write each rational a in the unique form $\frac{p}{q}$, where q is minimal positive; i.e., $q := \inf\{q' \in \mathbb{N}\colon q' > 0 \wedge (\exists p' \in \mathbb{Z}\colon aq' = p')\}$, and $p := aq$. (Lemma 9.7 makes this well defined and ensures that $p \in \mathbb{Z}$.) Then set

$$f(a) := \begin{cases} (2p, q) & \text{if } p \geqslant 0, \\ (1-2p, q) & \text{if } p < 0. \end{cases} \tag{12.17}$$

As $f(a)$ determine p and q uniquely, this defines an injection $f\colon \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$. Since Lemma 12.11 ensures that $\mathbb{N} \times \mathbb{N}$ is countable, so is $\mathbb{Q}$. $\square$

A similar reasoning shows:

Corollary 12.13 (AC) *Let $\{A_n \colon n \in \mathbb{N}\}$ be sets such that $\forall n \in \mathbb{N} \colon A_n$ is finite or countable. Then $\bigcup_{n\in\mathbb{N}} A_n$ is finite or countable.*

Proof. Corollary 12.10 along with the Axiom of Choice ensures existence of a collection $\{f_n \colon n \in \mathbb{N}\}$ of functions with $f_n \colon A_n \to \mathbb{N}$ an injection. (Note that a choice is made here which is why the Axiom of Choice needs to be invoked.) For each $a \in \bigcup_{n\in\mathbb{N}} A_n$, set

$$n(a) := \inf\{k \in \mathbb{N} \colon a \in A_k\}, \tag{12.18}$$

which exists by Lemma 9.7 and the fact that $a \in \bigcup_{n\in\mathbb{N}} A_n$ implies $\exists k \in \mathbb{N} \colon a \in A_k$. Then define a function $h\colon \bigcup_{n\in\mathbb{N}} A_n \to \mathbb{N} \times \mathbb{N}$ by

$$h(a) := \bigl(n(a), f_{n(a)}(a)\bigr). \tag{12.19}$$

Note that if $n(a) = n(b)$ and $f_{n(a)}(a) = f_{n(b)}(b)$, then the fact that $f_{n(a)} = f_{n(b)}$ is an injection forces $a = b$. It follows that h is an injection of $\bigcup_{n\in\mathbb{N}} A_n$ into $\mathbb{N} \times \mathbb{N}$ and so $\bigcup_{n\in\mathbb{N}} A_n$ is countable by Lemma 12.11. $\square$

In short, a countable union of countable sets is countable. Since A_n's are allowed to be empty, this includes finite unions as well. The need for Axiom of Choice is often circumvented by the sets A_n having some *a priori* structure.

One particular consequence of the argument in Corollary 12.13 (without needing the Axiom of Choice) is worth highlighting for future reference:

Lemma 12.14 *The set of algebraic numbers defined in* (11.20) *is countable.*

We leave a simple proof of this fact to a homework exercise. This statement appears to be first proved by Dedekind who communicated the proof in a letter to Cantor (the existence of which Cantor withheld in his main article on this topic).