11. Properties of the reals

Here we give some consequences of the construction of the reals. Among these are the definition of some basic functions; namely, roots, exponentials and logs. We also mention extensions of the field of reals to complex numbers and other, more esoteric, structures. Throughout we assume that a complete ordered field $(\mathbb{R}, +, 0, \cdot, 1, \leqslant)$ is given and use $\mathbb{N}$, resp., $\mathbb{Q}$ for the associated sets of naturals, resp., rationals in $\mathbb{R}$.

11.1 Archimedean property and density of (ir)rationals.

As our first consequence of the completeness of the reals, we extend the Archimedean property of the rationals to the reals. However, the proofs of these are very different.

Lemma 11.1 (Archimedean property of $\mathbb{R}$) $\forall x \in \mathbb{R}\colon x > 0 \Rightarrow (\exists n \in \mathbb{N}\colon x \cdot n > 1)$

Proof. Let $x > 0$ and suppose that $x \cdot n \leqslant 1$ for all $n \in \mathbb{N}$. This means that $\mathbb{N}$ is bounded by x^{-1} and so $\sup(\mathbb{N})$ exists by the least upper bound property. Then there is $n \in \mathbb{N}$ such that $n > \sup(\mathbb{N}) - 1$ because otherwise $\sup(\mathbb{N})$ is not the least upper bound. But then $\sup(\mathbb{N}) < n + 1$ which, since $n + 1 \in \mathbb{N}$, shows that $\sup(\mathbb{N})$ is not even an upper bound, a contradiction. □

The Archimedean property serves as a useful tool in proofs. For instance, it can be used to prove that rationals are spread "densely" in $\mathbb{R}$:

Lemma 11.2 (Density of rationals in $\mathbb{R}$)

$$\forall x, y \in \mathbb{R}\colon\ x < y \ \Rightarrow\ (\exists a \in \mathbb{Q}\colon x < a \wedge a < y) \tag{11.1}$$

Proof. We will suppose that $y > 0$ for otherwise we just replace y by $-x$ and x by $-y$ and then apply a sign change at the very end. The Archimedean principle tells us that there is $n \in \mathbb{N}$ such that $n(y - x) > 1$. Since $yn > 0$ the Archimedean principle also tells us that $A := \{k \in \mathbb{N}\colon k \geqslant yn\}$ is non-empty and so, by Lemma 9.7, $m := \inf(A)$ exists and obeys $m \in A$. But the latter forces $m - 1 < ny$ while the above shows

$$m - 1 \geqslant ny - 1 > ny - n(y - x) = nx \tag{11.2}$$

implying $m - 1 > nx$. Since $n > 0$, hereby we get $x < \frac{m-1}{n} < y$ as desired. □

The same actually applies to *irrationals*, which are those reals that are not rational.

Lemma 11.3 (Density of irrationals in $\mathbb{R}$)

$$\forall x, y \in \mathbb{R}\colon\ x < y \ \Rightarrow\ (\exists a \in \mathbb{R} \smallsetminus \mathbb{Q}\colon x < a \wedge a < y) \tag{11.3}$$

We leave the proof of this statement to a homework exercise.

It should be noted that the notion of "being dense" will be given another meaning once we discuss topological aspects of the reals. These considerations also drive Cantor's 1872 proof of existence of the reals.

We note that non-Archimedean extensions of $\mathbb{R}$ exist that are ordered fields; e.g., the so called *hyperreal numbers* or *surreal numbers*. The main feature of these is that, besides $\mathbb{R}$, they include infinitesimals (i.e., numbers those whose absolute value is smaller than any positive number) and also infinities (which are numbers that are arbitrarily large). These presence of infinitesimals makes these fields fail the least upper bound property (for otherwise they would be the same as $\mathbb{R}$) which has its disadvantages. But we can then talk about "infinitesimal increments" and other things that in $\mathbb{R}$ have to be dealt with via approximations. A variant of analysis, called *non-standard analysis*, is based on these fields instead of $\mathbb{R}$.

11.2 Roots, powers and logs.

Our construction of the reals was designed to fix one set of issues we had with the rationals; namely, the fact that non-empty bounded sets of rationals may not admit a supremum. However, we also pointed out algebraic deficiencies of rationals such as the absence of a rational solution to $x^2 = 2$. To see that this is fixed as well, we prove:

Theorem 11.4 (Arbitrary roots) *For each real $a \geqslant 0$ and natural $n \geqslant 2$ there exists a unique real $x \geqslant 0$ such that $x^n = a$.*

Proof. Fix a real $a > 0$ (if $a = 0$ then the claim is checked easily) and a natural $n \geqslant 1$ and let $A := \{y \geqslant 0 \colon y^n \leqslant a\}$. Then $0 \in A$, so $A \neq \emptyset$. Also the fact that $1 + a \geqslant 1$ gives $(1 + a)^n \geqslant 1 + a > a$ and so a simple argument by contradiction shows that $1 + a$ bounds all elements of A from above. Having shown that A is non-empty and admits an upper bound,

$$x := \sup(A) \tag{11.4}$$

is well defined. It remains to show that $x^n = a$.

We first check (not by induction but by invoking the distributive law and elementary manipulations with sums) that

$$\forall x, y \in \mathbb{R}\, \forall m \in \mathbb{N}\colon\ x^{m+1} - y^{m+1} = (x - y) \sum_{k=0}^{m} x^k y^{m-k} \tag{11.5}$$

Using that $0 \leqslant y < x$ gives $y^{m-k} \leqslant x^{m-k}$ for all terms in the sum, this shows

$$\forall x, y \in \mathbb{R}\, \forall \in \mathbb{N}\colon\ 0 \leqslant y \wedge y \leqslant x \Rightarrow x^n - y^n \leqslant n(x - y)x^n \tag{11.6}$$

where the case $n = 0$ is added for convenience and checked directly from the definition of n-th power. (We leave the details to homework.)

Now we finish the proof of the claim. First, since x is the supremum of A, for each $m \in \mathbb{N}$ there exists $y \in A$ such that $y \leqslant x \leqslant y + \frac{1}{m+1}$. (Indeed, otherwise $x - \frac{1}{m+1}$ is also an upper bound on A.) But this implies $x - y \leqslant \frac{1}{m+1}$ and so

$$x^n \leqslant y^n + n(x - y)x^n \leqslant a + n\frac{1}{m + 1}x^{n-1} \tag{11.7}$$

The Archimedean property now rules out that $x^n > a$ and so $x^n \leqslant a$. In order to show equality, suppose $x^n < a$. Then use the Archimedean property to find $\ell \in \mathbb{N}$ such that $(\ell + 1)(a - x^n) > n(2 + a)^{n-1}$ and set $y := x + \frac{1}{\ell+1}$. Since $x \leqslant 1 + a$, we then have $y \leqslant 1 + a + \frac{1}{\ell+1} \leqslant 2 + a$. Then (11.6) gives

$$y^n \geqslant x^n + n(y - x)y^{n-1} \leqslant x^n + \frac{n(2 + a)^{n-1}}{\ell + 1} < x^n + (a - x^n) = a \tag{11.8}$$

and so $y \in A$. But $x < y$ so this contradicts that x is the supremum of A. Hence we get $x^n = a$ as claimed.

The uniqueness of the solution to $x^n = a$ comes from the fact that if y is another real, then $x < y$ implies $x^n < y^n$ while $y < x$ implies $y^n < x^n$. $\square$

We will henceforth use notations

$$\sqrt[n]{a} \quad \text{or} \quad a^{1/n} \tag{11.9}$$

for the unique non-negative solution of $x^n = a$ for $a \geqslant 0$.

Recall that integer powers are defined from natural powers via $a^{-n} := (a^{-1})^n$ whenever $a > 0$ and $n \in \mathbb{N}$. The natural roots then allow us to define symbols of the form $(a^p)^{1/q}$ and $(a^{1/q})^p$ for any $p, q \in \mathbb{Z}$ with $q \neq 0$. It turns out that these quantities only depend on the value of p/q. This leads to:

Theorem 11.5 (Exponential) *Given $a > 0$ and $p, q, p', q' \in \mathbb{Z}$ such that $q, q' \neq 0$, we have*

$$(a^p)^{1/q} = (a^{1/q})^p \tag{11.10}$$

and

$$(a^p)^{1/q} = (a^{p'})^{1/q'} \tag{11.11}$$

Denoting by $a^{p/q}$ the common value of these quantities, for each $x \in \mathbb{R}$ we then set

$$a^x := \begin{cases} \sup\{a^z \in \mathbb{R} \colon z \in \mathbb{Q} \wedge z \leqslant x\}, & \text{if } a > 1, \\ \inf\{a^z \in \mathbb{R} \colon z \in \mathbb{Q} \wedge z \leqslant x\}, & \text{if } a < 1, \end{cases} \tag{11.12}$$

and put $1^x := 1$. We then have

$$\forall a > 0\, \forall x, y \in \mathbb{R} \colon a^{x+y} = a^x \cdot a^y \wedge a^{x \cdot y} = (a^x)^y = (a^y)^x \tag{11.13}$$

Also, $\forall a > 0\, \forall x \in \mathbb{R} \colon a^x > 0$.

We call the function $x \mapsto a^x$ the *exponential* of x of *base a*. The proof of this theorem is relegated to a homework exercise. Once the exponential is defined, we can then prove:

Theorem 11.6 (Logarithm) *For each $a > 0$ with $a \neq 1$ and $x > 0$ there exists a unique $y \in \mathbb{R}$ such that $a^y = x$.*

A proof of this theorem also comes in a homework exercise. We call the unique y with the above property the *logarithm base a of x* with the notation $\log_a(x)$. We thus have

$$\forall a > 0 \forall x > 0 \colon\ a^{\log_a(x)} = x \tag{11.14}$$

The properties in (11.13) then readily give:

Lemma 11.7 *Let $a > 0$ be such that $a \neq 1$. Then*

$$\forall x, y > 0 \colon\ \log_a(x \cdot y) = \log_a(x) + \log_a(y) \tag{11.15}$$

and

$$\forall x > 0 \forall y \in \mathbb{R} \colon\ \log_a(x^y) = y \log_a(x) \tag{11.16}$$

We leave this exercise to a homework exercise as well. We also note that both the exponential and logarithm functions can and will be constructed (and thus defined) using methods of calculus. These definitions will naturally coincide with those above.

11.3 Beyond the reals.

As we noted after the proof of Theorem 10.17, the fact that there is only one system of reals modulo order-preserving isomorphism implies that there is only one real analysis one can build out of Zermelo's axioms. Still, the reader might wonder whether other natural fields exist that are of significance for analysis. A very important extension is that to *complex numbers* discovered by G. Cardano in his solution of the cubics. Formally, the complex numbers are defined as a vector space over $\mathbb{R}$ with basis $\{1, \mathrm{i}\}$, i.e.,

$$\mathbb{C} := \{x + \mathrm{i}y \colon x, y \in \mathbb{R}\} \tag{11.17}$$

with multiplication extended to the *imaginary number* i via

$$\mathrm{i} \cdot \mathrm{i} = -1, \tag{11.18}$$

form a field.

The field of complex numbers is an extension of $\mathbb{R}$ with the property that all polynomials, even those with coefficients in $\mathbb{C}$, have a root in $\mathbb{C}$ and thus factor completely. This means that $\mathbb{C}$ is *algebraically closed*. A deficiency of $\mathbb{C}$ over $\mathbb{R}$ is that (as you have been asked to show in homework) it does not admit an ordering that would make it an ordered field (in the sense of Definition 7.2).

It should be noted that no finite field is algebraically closed. This is because the polynomial

$$P(x) := 1 + \prod_{i=0}^{n} (x - z_i), \tag{11.19}$$

where $z_0, \dots, z_n$ are the elements of the field, is never zero. The reals are not algebraically closed either (as $x^2 + 1 = 0$ has no real roots) nor is its subfield of real *algebraic numbers*,

$$\Bigl\{x \in \mathbb{R} \colon \bigl(\exists n \in \mathbb{N}\, \exists a_0, \dots, a_n \in \mathbb{Q} \colon a_n \neq 0 \,\wedge\, a_n x^n + \dots + a_1 x + a_0 = 0\bigr)\Bigr\}. \tag{11.20}$$

(That this is a field requires showing that the sum and product of algebraic numbers is algebraic.) However, the field of complex algebraic numbers (defined the same way as above with just $x \in \mathbb{C}$ instead) is algebraically closed. Still, for reasons mentioned numerous times above, the field of complex algebraic numbers is too small for analysis.