10. The reals via Dedekind cuts

We now move towards the axiomatic definition (and construction) of the set of real numbers. Not all details will be spelled out; we refer to the aforementioned textbook by Yannis Moschovakis for more details.

10.1 Dedekind cuts.

As noted above, the rationals lack the property that some (even simple) bounded sets fail to admit a supremum and an infimum. In order to formalize this better, we start with the following concept:

Definition 10.1 *An ordered field* $(F, +, 0, \cdot, 1, \leqslant)$ *is said to be* complete *if every non-empty subset thereof that admits an upper bound admits a supremum.*

The reader might wonder why we are not saying anything about infima of sets that admit a lower bound. This is because for ordered fields we have:

Lemma 10.2 *Let* $(F, +, 0, \cdot, 1, \leqslant)$ *be a complete ordered field. Then any* $A \subseteq F$ *that admits a lower bound admits an infimum and, in fact,*

$$\inf(A) = -\sup(-A) \tag{10.1}$$

where

$$-A := \{a \in F\colon\ -a \in A\} \tag{10.2}$$

admits an upper bound and thus a supremum.

Proof. The claim hinges on the following fact

$$\forall x \in F\colon\ x \in F \text{ is a lower bound on } A \Leftrightarrow -x \text{ is an upper bound on } -A \tag{10.3}$$

whose proof we leave to the reader. Using that the field is complete and A admits a lower bound, (10.3) implies that $\sup(-A)$ exists and $-\sup(-A)$ is a lower bound on A. Moreover, if x is another lower bound on A, then (10.3) and the definition of supremum forces $\sup(A) \leqslant -x$ which rewrites into $x \leqslant -\sup(-A)$. Hence, $-\sup(-A)$ is the greatest lower bound on A, and thus the infimum of A. □

We now axiomatize the real numbers as follows:

Definition 10.3 (Real numbers) *A system of reals is any complete ordered field.*

A natural question is whether any such object exists. This is the content of:

Theorem 10.4 (Dedekind 1872, Cantor 1872) *There exists at least one system of reals.*

In order to prove Theorem 10.4, we will follow a construction that represents the real numbers by particular sets of rationals. Since all systems of rationals are isomorphic, we pick one and denote it $(\mathbb{Q}, +, 0, \cdot, 1, \leqslant)$. We then put forward:

Definition 10.5 (Dedekind cut) *We say that a set* $A \subseteq \mathbb{Q}$ *is a (Dedekind) cut if*

(C1) $A \neq \emptyset \ \wedge \mathbb{Q} \smallsetminus A \neq \emptyset$,
(C2) $\forall a, b \in \mathbb{Q}\colon a \in A \ \wedge\ b \leqslant a \Rightarrow b \in A$,
(C3) $\forall a \in A\, \exists b \in A\colon\ a < b$.

We write

$$\mathbb{R} := \{A \subseteq \mathbb{Q} \colon \text{C1-C3 hold}\} \tag{10.4}$$

for the set of all cuts.

These defining properties C1-C3 may be verbalized as follows: C1 means that the pair $(A, \mathbb{Q} \smallsetminus A)$ forms a non-trivial partition of $\mathbb{Q}$, C2 means that A is an interval which, by C3 contains no largest element. It is easy to check that

$$\forall a \in \mathbb{Q} \colon \ \{b \in \mathbb{Q} \colon b < a\} \in \mathbb{R} \tag{10.5}$$

and

$$\{b \in \mathbb{Q} \colon b < 0 \ \vee b^2 < 2\} \in \mathbb{R} \tag{10.6}$$

In fact, as formalized in the next lemma, all cuts look like like this:

Lemma 10.6 *For all $A \in \mathbb{R}$,*

(1) $\forall a \in \mathbb{Q} \smallsetminus A \, \forall b \in \mathbb{Q} \colon a \leqslant b \Rightarrow b \in \mathbb{Q} \smallsetminus A$,
(2) $\forall a \in A \, \forall b \in \mathbb{Q} \smallsetminus A \colon a < b$

In addition, we have

$$\forall A \in \mathbb{R} \colon \ \{b - a \colon a \in A \wedge b \in \mathbb{Q} \smallsetminus A\} = \{c \in \mathbb{Q} \colon c > 0\} \tag{10.7}$$

Proof. Properties (1) and (2) follow directly from C1-C3. For (10.7), $\subseteq$ follows from (2). For the opposite inclusion, assume that the set on the left misses a point $c \in \mathbb{Q}$ with $c > 0$. For all $a \in A$ we then have $a + c \notin \mathbb{Q} \smallsetminus A$ and thus $a + c \in A$. Induction then shows that $a + cn \in A$ for all $n \in \mathbb{N}$. But then $A = \mathbb{Q}$ for otherwise there is $x \in \mathbb{Q} \smallsetminus A$ which, by C2, must obey $a + cn \leqslant x$ for all $n \in \mathbb{N}$ which is impossible by the Archimedean principle. But $A = \mathbb{Q}$ is impossible by C1 either and so no such c exists to begin with. □

10.2 Ordering relation for cuts.

In order to turn $\mathbb{R}$ into a system of reals, we need to define addition $\oplus$, multiplication $\odot$ and an ordering relation $\preccurlyeq$ so that $\mathbb{R}$ becomes an ordered field. We start by the ordering relation. Define

$$\forall A, B \in \mathbb{R} \colon \ A \preccurlyeq B := \ A \subseteq B \tag{10.8}$$

We then have:

Lemma 10.7 *$\preccurlyeq$ is a total ordering of $\mathbb{R}$.*

Proof. That $\preccurlyeq$ is a partial order follows from that being true about $\subseteq$; see Lemma 3.4. It remains show that the relation is connex; meaning that $\forall A, B \in \mathbb{R} \colon A \preccurlyeq B \vee B \preccurlyeq A$. Assume for contradiction that this fails for some $A, B \in \mathbb{R}$. Then $A \smallsetminus B \neq \emptyset$ and $B \smallsetminus A \neq \emptyset$ which implies existence of $a \in A \smallsetminus B$ and $b \in B \smallsetminus A$. The total ordering of $\mathbb{Q}$ gives that one of $a = b$, $a < b$ or $b < a$ are TRUE. Equality is ruled out directly from $a \notin B$ and $b \in B$. If $a < b$ is TRUE, then $b \in B$ and C2 forces $a \in B$, a contradiction. The case $b < a$ is handled by symmetry. Hence, $\preccurlyeq$ is connex and the ordering is total. □

Next we check that the ordering $\preccurlyeq$ has the least upper bound property:

Lemma 10.8 *We have*

$$\forall C \subseteq \mathbb{R}\colon\ C \neq \emptyset \ \wedge\ \bigl(\exists B \in \mathbb{R} \forall A \in C\colon\ A \subseteq B\bigr) \Rightarrow \bigcup C \in \mathbb{R}. \tag{10.9}$$

In particular, each non-empty $C \subseteq \mathbb{R}$ that admits an upper bound admits a supremum.

Proof. Let $C \subseteq \mathbb{R}$ be such that, for some $B \in \mathbb{R}$ we have $\forall A \in C\colon\ A \subseteq B$. We claim that

$$\sup(C) = \bigcup C = \bigcup_{A \in C} A. \tag{10.10}$$

The proof comes in two parts.

Part 1: $\bigcup C$ is a cut: We need to verify C1-C3 for $\bigcup C$. Starting with C1, note that $\bigcup C \subseteq B$ and so, by C1 for B, we have $\mathbb{Q} \smallsetminus \bigcup C \neq \emptyset$. Since $C \neq \emptyset$ there is $A \in C$ and so $A \subseteq \bigcup C$. Hence, by C1 for A we get $\bigcup C \neq \emptyset$ showing that $\bigcup C$ obeys C1.

The argument is routine for C2-C3. Indeed, let $a \in \bigcup C$ and let $b \leqslant a$. Then there is $A \in C$ such that $a \in A$ and, by C2 for A, we have $b \in A$ and thus $b \in \bigcup C$, showing that $\bigcup C$ obeys C2. Similarly, if $a \in \bigcup C$, then for some $A \in C$ we have $a \in A$ and, by C3 for A, there is $b \in A$ with $a < b$. But then $b \in \bigcup C$ showing that $\bigcup C$ obeys C3 as well.

Part 2: $\bigcup C$ is the supremum of C with respect to $\leqslant$: Here we first note that $\bigcup C$ subsumes all $A \in C$ and is thus an upper bound on C. If B is another such upper bound, then B subsumes all $A \in C$ and so it subsumes $\bigcup C$ as well. This shows that $\bigcup C$ is the least upper bound on C and is thus the supremum of C. □

10.3 Addition for cuts.

Next we move to the operation of addition on $\mathbb{R}$, to be denoted by symbol $\oplus$ in order to prevent confusion with the operation of addition in $\mathbb{Q}$. Given any sets $A, B \subseteq \mathbb{Q}$, define the following sets of rationals

$$\begin{aligned} A \oplus B &:= \bigl\{a + b\colon a \in A \wedge b \in B\bigr\} \\ \underline{0} &:= \{a \in \mathbb{Q}\colon a < 0\} \\ \ominus A &:= \bigl\{a \in \mathbb{Q}\colon (\exists b \in \mathbb{Q} \smallsetminus A\colon a + b < 0)\bigr\} \end{aligned} \tag{10.11}$$

The first point to check is that, for A and B cuts, the sets $A \oplus B$ and $\ominus A$ are also cuts. (For $\underline{0}$ this was done in (10.5).) We start with the former:

Lemma 10.9 $\forall A, B \in \mathbb{R}\colon\ A \oplus B \in \mathbb{R}$

Proof. From $A, B \neq \emptyset$ we get $A \oplus B \neq \emptyset$. On the other hand, if $a' \in \mathbb{Q} \smallsetminus A$ and $b' \in \mathbb{Q} \smallsetminus B$, then $a < a'$ for all $a \in A$ and $b < b'$ for all $b \in B$ by Lemma 10.6. It follows that $a' + b' \notin A \oplus B$ and so $A \oplus B \neq \mathbb{Q}$. We have proved C1 for $A \oplus B$.

For C2, if $a + b \in A \oplus B$ and $c \in \mathbb{Q}$ obeys $c \leqslant a + b$, then $a' := c - b$ obeys $a' \leqslant a$. Hence $a' \in A$ by C2 for A and so $c = a' + b \in A \oplus B$, proving C2 for $A \oplus B$. Concerning C3, if $a + b \in A \oplus B$ for some $a \in A$ and $b \in B$ then C2 for A implies existence of $a' \in A$ with $a < a'$. This gives $a' + b \in A \oplus B$ with $a + b < a' + b$, proving C3 for $A \oplus B$ as well. □

For the operation $\ominus$ we get even a bit more:

Lemma 10.10 $\forall A \in \mathbb{R}\colon\ \ominus A \in \mathbb{R} \ \wedge\ A \oplus (\ominus A) = \underline{0}$

Proof. Let $A \in \mathbb{R}$. First note that, if $c \in A$ then for all $b \in \mathbb{Q} \smallsetminus A$, Lemma 10.6(2) gives $b - c > 0$ showing that $-c \notin \ominus A$. This is worthy of noting separately,

$$\forall c \in \mathbb{Q}\colon\ c \in A \ \Rightarrow\ -c \notin \ominus A \tag{10.12}$$

From $A \neq \emptyset$, we then get $\ominus A \neq \mathbb{Q}$. Next, if $b \in \mathbb{Q} \smallsetminus A$ and $d \in \mathbb{Q}$ obeys $d > b$, then $b + (-d) < 0$ and so $-d \in \ominus A$, thus showing C1 for $\ominus A$. Conditions C2 and C3 are then checked directly from the definition: if $a \in \mathbb{Q}$ is such that $a + b < 0$ for some $b \in \mathbb{Q} \smallsetminus A$, then $a' + b < 0$ for all $a' \leqslant a$ and, by the Archimedean principle, there is $a'' > a$ such that $a'' + b < 0$ still holds. It follows that $\ominus A \in \mathbb{R}$ as desired.

Moving to the proof of $A \oplus (\ominus A) = \underline{0}$, assume first that, for some $a \in A$ and $b \in \ominus A$ we have $a + b > 0$. Then $-a - b < 0$ and, since the contrapositive of (10.12) forces $-b \notin A$, we get $-a \in \ominus A$. But (10.12) then forces $a \notin A$, a contradiction. As $0 \notin A \oplus (\ominus A)$ by (10.12) again, we have proved $A \oplus (\ominus A) \subseteq \underline{0}$.

In order to prove the opposite inclusion, let $c \in \underline{0}$. As $c < 0$, (10.7) ensures existence of $a \in A$ and $b \in \mathbb{Q} \smallsetminus A$ such that $b - a = -c/2$. But then $b + (-a + c) = c/2 < 0$ and so $-a + c \in \ominus A$. This shows $c = a + (-a + c) \in A \oplus (\ominus A)$, thus proving $\underline{0} \subseteq A \oplus (\ominus A)$. □

Using these lemmas we conclude:

Corollary 10.11 *$\oplus$ is a commutative and associative binary operation (of addition) on $\mathbb{R}$ with $\underline{0}$ being the zero element and $\ominus A$ being the inverse element to A. In short, $(\mathbb{R}, \oplus)$ is a commutative group with unit element $\underline{0}$.*

Proof. Commutativity and associativity is checked readily from the definition of $A \oplus B$. That $A \oplus \underline{0} = A$ for each $A \in \mathbb{R}$ is checked directly from the definition of the cut. The inverse element property was proved in Lemma 10.10. □

It remains to link addition to the ordering relation, and thus show that $\underline{0} \preccurlyeq A$ is equivalent to $\ominus A \preccurlyeq \underline{0}$:

Lemma 10.12 $\forall A, B, C \in \mathbb{R}\colon\ A \preccurlyeq B \Rightarrow A \oplus C \preccurlyeq B \oplus C$

Proof. Let A, B, C be cuts. Then $A \preccurlyeq B$ means $A \subseteq B$. The definition of addition then gives $A \oplus C \subseteq B \oplus C$ and so $A \oplus C \preccurlyeq B \oplus C$. □

10.4 Multiplication for cuts.

The multiplication between cuts is defined similarly, albeit in two stages. Writing $A \prec B$ for $A \preccurlyeq B \wedge A \neq B$ and abbreviating $\mathbb{R}^- := \{A \in \mathbb{R}\colon A \prec \underline{0}\}$, we first set

$$\forall A, B \in \mathbb{R}^-\colon\quad A \odot B := \ominus\big\{-a \cdot b\colon\ a \in A \wedge b \in B\big\}. \tag{10.13}$$

Before we proceed to other cases, we need to check that the resulting object is a cut.

Lemma 10.13 $\forall A, B \in \mathbb{R}^-\colon A \odot B \in \mathbb{R}$

Proof. Given $A, B \in \mathbb{R}^-$ let $C := \{-a \cdot b\colon\ a \in A \wedge b \in B\}$. Since A and B contain only negative rationals, for each $a \in A$, the term $-a \cdot b$ increases if b increases, and decreases if b does whenever $b \in B$. This implies C2 and C3 for C. For C1 we note that $C \neq \emptyset$ because there is at least one $a \in A$ and one $b \in B$. Since $a \cdot b \neq 0$ for all $a \in A$ and $b \in B$ we also have $C \neq \mathbb{Q}$. This shows $C \in \mathbb{R}$ and, by Lemma 10.10, also $A \odot B \in \mathbb{R}$. □

We now complete the definition of $\odot$ by setting

$$A \odot B := \begin{cases} (\ominus A) \odot (\ominus B) & \text{if } \underline{0} < A \wedge \underline{0} < B, \\ \ominus\big((\ominus A) \odot B\big) & \text{if } \underline{0} < A \wedge B < \underline{0} \\ \ominus\big(A \odot (\ominus B)\big) & \text{if } A < \underline{0} \wedge \underline{0} < B \\ \underline{0} & \text{if } A = \underline{0} \vee B = \underline{0} \end{cases} \tag{10.14}$$

and add the following objects:

$$\underline{1} := \{a \in \mathbb{Q} \colon a < 1\} \tag{10.15}$$

and

$$\forall A \in \mathbb{R} \smallsetminus \{\underline{0}\} \colon \quad A^{-1} := \begin{cases} \big\{b \in \mathbb{Q} \colon (\forall a \in A \colon 1 < a \cdot b)\big\}, & \text{if } A < \underline{0}, \\ \ominus(\ominus A)^{-1}, & \text{if } 0 < A. \end{cases} \tag{10.16}$$

Following similar arguments as for addition, the reader will again check:

Lemma 10.14 $\forall A \in \mathbb{R} \colon A \neq \underline{0} \Rightarrow A^{-1} \in \mathbb{R} \wedge A \odot A^{-1} = \underline{1}$

Lemma 10.15 *$\odot$ is commutative, associative and distributive about $\oplus$. In addition, $\underline{1}$ is the unit element and A^{-1} is the inverse element for each $A \in \mathbb{R}$ with $A \neq \underline{0}$. In short,*

$$(\mathbb{R}, \oplus, \underline{0}, \odot, \underline{1}) \tag{10.17}$$

is a field.

Lemma 10.16 $\forall A, B, C \in \mathbb{R} \colon (A \leqslant B \wedge \underline{0} \leqslant C) \Rightarrow A \odot C \leqslant B \odot C.$

Leaving the proof these claims to an exercise, we are now ready to give:

Proof of Theorem 10.4, existence. Combining of Lemma 10.7, Corollary 10.11, Lemma 10.12 and Lemmas 10.13-10.16, $(\mathbb{R}, \oplus, \underline{0}, \odot, \underline{1}, \leqslant)$ is an ordered field. The ordering is complete by Lemma 10.8 and so $(\mathbb{R}, \oplus, \underline{0}, \odot, \underline{1}, \leqslant)$ is a system of reals. □

Another way to construct the reals follows an argument of G. Cantor which is in fact more general than the problem itself. The argument goes by interpreting $\mathbb{Q}$ as a metric space endowed with the metric $\varrho(a, b) := |a - b|$. The set of reals is the identified with equivalence classes of Cauchy sequences in $(\mathbb{Q}, \varrho)$. We will return to this argument when we discuss metric spaces in detail.

10.5 Uniqueness.

Having established existence, we now move to the statement and proof of uniqueness.

Theorem 10.17 (Uniqueness) *Let $(\mathbb{R}, \oplus, \underline{0}, \odot, \underline{1}, \leqslant)$ be as above. For any complete ordered field $(F, +, 0, \cdot, 1, \leqslant)$, there is a bijection $\phi \colon \mathbb{R} \to F$ which is an order-preserving (field) isomorphism. (The latter means that ϕ obeys properties (1-4) in Theorem 7.7).*

Proof (main steps). Let $(F, +, 0, \cdot, 1, \leqslant)$ be complete ordered field. First we identify elements of F with Dedekind cuts. For this let $\mathbb{N}_F$ be the naturals of F (see (7.8)) and write

$$\mathbb{Q}_F := \big\{r^{-1} \cdot (m - n) \colon m, n, r \in \mathbb{N}_F \wedge r \neq 0\big\} \tag{10.18}$$

for the *rationals of* F. Following the proof of Theorem 10.4, we now construct a complete ordered field $(\mathbb{R}_F, \oplus_F, 0_F, \odot_F, 1_F, \preccurlyeq_F)$ of Dedekind cuts based on rationals $\mathbb{Q}_F$.

Next observe that, since F has the least upper bound property the cuts in $\mathbb{R}_F$ do admit a universal representation:

Lemma 10.18 *We have:*

$$\forall A \in \mathbb{R}_F\colon \quad \sup(A) \text{ exists } \wedge \ A = \{a \in \mathbb{Q}_F\colon a < \sup(A)\} \tag{10.19}$$

In particular, $\sup\colon \mathbb{R}_F \to F$ *is a bijection.*

Now check that, for all $A, B \in \mathbb{R}_F$,

$$\begin{aligned} \sup(A \oplus_F B) &= \sup(A) + \sup(B) \\ \sup(A \odot_F B) &= \sup(A) \cdot \sup(B) \\ \sup(\ominus_F A) &= -\sup(A) \end{aligned} \tag{10.20}$$

and, if $A \neq 0_F$, also

$$\sup(A^{-1}) = \sup(A)^{-1} \tag{10.21}$$

Noting that $\sup(0_F) = 0$ and $\sup(1_F) = 1$, and that $A \preccurlyeq_F B$ is equivalent to $\sup(A) \leqslant \sup(B)$, the map $\sup\colon \mathbb{R}_F \to \mathbb{R}$ is an order preserving isomorphism.

We now recall that Theorem 7.7 asserts the existence of a bijection $\psi\colon \mathbb{Q} \to \mathbb{Q}_F$ which is an order-preserving isomorphism. The image map associated with ψ acting as

$$\psi(A) := \{\psi(x) \in F\colon x \in A\} \tag{10.22}$$

then maps cuts over $\mathbb{Q}$ to those over $\mathbb{Q}_F$ and thus defines a bijection $\psi\colon \mathbb{R} \to \mathbb{R}_F$. Since all arithmetic operations on cuts are defined the same way in $\mathbb{R}$ as in $\mathbb{R}_F$, we get that that ψ is an order preserving isomorphism. Setting

$$\phi := \sup \circ \psi \tag{10.23}$$

we get the map in the claim. □

A take-home message of this section is that the reals exist as a complete ordered field and they are unique modulo order-preserving isomorphism. The latter implies that there is only one real analysis one can build out of Zermelo's axioms.