HW#9: due Thr 3/16/2023 (10AM)

This exercise practices completeness and compactness. The last two problems are on upper/lower limits of sequences. Problem #1 offers additional practice of convergence of sequences.

Problem 1: Given two reals $a_0, b_0 > 0$, define $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ recursively so that

$$\forall n \in \mathbb{N}\colon\ a_{n+1} = \frac{a_n + b_n}{2} \ \wedge\ b_{n+1} = \sqrt{a_n \cdot b_n}$$

Prove that $a_\infty := \lim_{n\to\infty} a_n$ and $b_\infty := \lim_{n\to\infty} b_n$ exist and $a_\infty = b_\infty$. Justify *all* steps.

Problem 2: Let $\mathbb{R}_+ := \{x \in \mathbb{R}\colon x \geqslant 0\}$ and, given a natural $n \geqslant 1$, define $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(x) := 2 + \frac{1}{1 + x^n}$$

Prove that there exists a unique $x \in \mathbb{R}_+$ with $f(x) = x$. Give detailed proofs of all claims that we have not proved in class (which means: don't rely on those we have not proved).

Problem 3: (RUDIN) PAGE 43, EX 13 (Compact set with countably many limit points.)

Problem 4: (RUDIN) PAGE 43, EX 14 (Open cover of $(0, 1)$ with no finite subcover.)

Problem 5: (RUDIN) PAGE 44, EX 17 (Reals with decimal expansion in 4s and 7s.)

Problem 6: (RUDIN) PAGE 45, EX 23 (A separable metric space has a countable base.)

Problem 7: (RUDIN) PAGE 45, EX 25 (A compact space is separable.)

Problem 8: (RUDIN) PAGE 45, EX 26 (A metric space whose each infinite subset admits a limit point is compact.)

Problem 9: (RUDIN) PAGE 78, EX 4 (Upper/lower limits)

Problem 10: (RUDIN) PAGE 78, EX 5 (*Limes superior* of sum of two sequences)