HW#8: due Fri 3/10/2023 (6PM)

This exercise practices metric spaces; specifically, Cauchy and convergent sequence and subsequences and the notions from point set topology such as interior, closure and topological boundary. The last three problems deal with complete metric spaces. Problem #6 is laborious but it is important that you write a detailed solution as annotated.

Problem 1: Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X,ϱ). Prove that the following are equivalent:

(1) $\{x_n\}_{n\in\mathbb{N}}$ is convergent,
(2) there exists $z \in X$ such that every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ contains a subsequence that converges to z.

Problem 2: Let (X,ϱ) be a metric space. Prove that for all sets $A, B \subseteq X$ the following holds:

$$A \subseteq B \quad\Rightarrow\quad \text{int}(A) \subseteq \text{int}(B) \quad\wedge\quad \overline{A} \subseteq \overline{B}$$

and

$$X \smallsetminus \text{int}(A) = \overline{X \smallsetminus A} \quad\wedge\quad X \smallsetminus \overline{A} = \text{int}(X \smallsetminus A)$$

In particular,

$$\partial A = \overline{A} \cap \overline{X \smallsetminus A} = \partial(X \smallsetminus A)$$

The interior $\text{int}(A)$, the closure $\overline{A}$ and the boundary ∂A are defined as in class. (All of these can be proved using the topological axiomatization of open sets.)

Problem 3: Let $\mathbb{R}$ be endowed with the Euclidean metric and note that both $\mathbb{N}$ and $\mathbb{Q}$ are subsets of $\mathbb{R}$ (which makes relative notions possible). Do as follows:

(1) Give an example of a set $A \subseteq \mathbb{Q}$ that is relatively open in $\mathbb{Q}$ but not open in $\mathbb{R}$.
(2) Prove that every subset $A \subseteq \mathbb{N}$ is relatively open and relatively closed.
(3) Prove that all subsets of $\mathbb{N}$ are closed in $\mathbb{R}$.
(4) Prove that if $A \subseteq \mathbb{Q}$ is both relatively open and relatively closed, then it is a finite or countable union of disjoint intervals (in $\mathbb{Q}$) with irrational endpoints.

Problem 4: (RUDIN) PAGE 82, EX 21 (Each nested sequence of shrinking sets in a complete space has a point in common.) Then show, by way of a counterexample, that the restriction on diameters cannot be left out.

Problem 5: (RUDIN) PAGE 82, EX 22 (Baire's theorem: A sequence of open dense sets in a complete space has $\neq \varnothing$ intersection.)

Problem 6: (RUDIN) PAGE 82, EX 24 (Completion of a metric space)