HW#7: due Thr 3/2/2023

This exercise practices metric spaces and metric space convergence, as well as terms from metric-space topology: open and closed sets, interior and closure, etc. There are 7 problems total.

Problem 1: For $k \geqslant 1$ and $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ define

$$\varrho_\infty(x, y) := \max_{i=1,\dots,k} |x_i - y_i|$$

Do as follows:

(1) Prove that ϱ_∞ is a metric on $\mathbb{R}^k$.

(2) Prove that if $\{x^{(n)}\}_{n\in\mathbb{N}}$ is a sequence in $(\mathbb{R}^k, \varrho_\infty)$ and $x_i^{(n)}$ is the ith coordinate of $x^{(n)}$, then

$$\{x^{(n)}\}_{n\in\mathbb{N}} \text{ converges} \quad \Leftrightarrow \quad \forall i = 1, \dots, k\colon \quad \lim_{n\to\infty} x_i^{(n)} \text{ exists (in } \mathbb{R})$$

Problem 2: Given a metric space (X, ϱ), recall that $B(x, r) := \{y \in X : \varrho(x, y) < r\}$ is the open ball of radius r centered at x. Define $B'(x, r) := \{y \in X : \varrho(x, y) \leqslant r\}$ be the *closed ball* of radius r centered at x. Prove that

$$X \smallsetminus B'(x, r) \text{ is open}$$

and so $B'(x, r)$ is closed (justifying its name). Then give an example of a metric space and an open/closed ball such that the closure of $B(x, r)$ is not $B'(x, r)$, i.e.,

$$\overline{B(x, r)} \neq B'(x, r)$$

Problem 3: An ultrametic on X is a metric ϱ on X such that

$$\forall x, y, z \in X\colon \qquad \varrho(x, y) \leqslant \max\{\varrho(x, z), \varrho(y, z)\}.$$

Prove that, in this metric, every open ball $B(x, r) := \{y \in X : \varrho(x, y) < r\}$ is closed and every closed ball $B'(x, r) := \{y \in X : \varrho(x, y) \leqslant r\}$ is open. Determine the topological boundary $\partial B(x, r)$ of $B(x, r)$.

To give an example of such a setting, let $X := \{0, 1\}^{\mathbb{N}}$ be the set of all zero-one valued sequences. Prove that then $\varrho\colon X \times X \to \mathbb{R}$ defined for $\sigma, \sigma' \in X$ with $\sigma \neq \sigma'$ by

$$\varrho(\sigma, \sigma') := 2^{-\inf\{k\in\mathbb{N}\colon \sigma(k)\neq\sigma'(k)\}}$$

(and by $\varrho(\sigma, \sigma) := 0$) is an ultrametric.

Problem 4: Let (X, ϱ) be a metric space. Prove that for any two X-valued sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$,

$$\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \text{ Cauchy} \;\Rightarrow\; \lim_{n\to\infty} \varrho(x_n, y_n) \text{ exists in } \mathbb{R}$$

Here the limit on the right is with respect to the Euclidean distance on $\mathbb{R}$.

Problem 5: (RUDIN) PAGE 43, EX 9

Problem 6: (RUDIN) PAGE 45, EX 22

Problem 7: (RUDIN) PAGE 45, EX 29