HW#6: due Thr 2/23/2023

This assignment practices cardinality, countability, uncountability, etc. A few problems on metric spaces and metric space convergence are added to get started on the subject.

Problem 1: Prove that there exists an injection $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Conclude that all Euclidean spaces are of cardinality of the continuum (meaning: are equinumerous to $\mathbb{R}$). *Hint:* Observe that Dedekind cuts give an injection of $\mathbb{R}\to\mathcal{P}(\mathbb{Q})$.

Problem 2: Prove that the set $\{0,1\}^{\mathbb{R}}$ of functions $\mathbb{R}\to\{0,1\}$ cannot be labeled by the reals. More precisely, prove that there is no surjection $f\colon\mathbb{R}\to\{0,1\}^{\mathbb{R}}$. *Hint:* Use the Cantor diagonal argument.

Problem 3: (RUDIN) PAGE 43, EX 2-3 (Existence of non-algebraic reals.)

Problem 4: (RUDIN) PAGE 44, EX 11 (Various metrics and non-metrics on $\mathbb{R}$)

Problem 5: Define a sequence $\{a_n\}_{n\in\mathbb{N}}$ recursively as

$$a_0 := 1 \;\wedge\; \forall n\in\mathbb{N}\colon\; a_{n+1} := \frac{1}{1+a_n}$$

Prove the following:

(1) $\{a_{2n}\}_{n\in\mathbb{N}}$ is decrasing
(2) $\{a_{2n+1}\}_{n\in\mathbb{N}}$ is increasing
(3) $\forall m,n\in\mathbb{N}\colon a_{2m+1}\leqslant a_{2n}$
(4) $L := \inf_{n\in\mathbb{N}} a_{2n} = \sup_{n\in\mathbb{N}} a_{2n+1}$
(5) $\lim_{n\to\infty} a_n$ exists and equals L.
(6) Compute L and show that $L\notin\mathbb{Q}$.

The point of this exercise is, beyond practice of limits and monotonicity, that a sequence of rationals can be Cauchy yet not convergent in $\mathbb{Q}$.

Problem 6: For any $x,y\in\mathbb{R}$ let $d(x,y) := |\frac{y}{1+|y|}-\frac{x}{1+|x|}|$. Do the following:

(1) Prove that d is a metric on $\mathbb{R}$.
(2) Letting $x_n := n$, prove that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy with no limit in $\mathbb{R}$.

(Note that $\{x_n\}_{n\in\mathbb{N}}$ is NOT Cauchy under the Euclidean metric.)

Problem 7: (RUDIN) PAGE 78, EX 2

Problem 8: (RUDIN) PAGE 82, EX 20

Problem 9: Prove the following fact about real numbers

$$\forall x,y\in\mathbb{R}\colon\; x,y>0 \;\wedge\; x<1 \;\Rightarrow\; \exists n\in\mathbb{N}\colon x^n<y$$

This is useful when proving convergence of exponentially decaying sequences.
