HW#5: due Thr 2/16/2023

This exercise practices the reals and facts about cardinality.

Problem 1: Let $(F, +, 0, \cdot, 1, \leqslant)$ be a complete ordered field, $\mathbb{N}_F$ the naturals of F and

$$\mathbb{Q}_F := \left\{ r^{-1} \cdot (m - n) \colon m, n, r \in \mathbb{N}_F \wedge r \neq 0 \right\}$$

the rationals of F. This permits us to define the concept of a (Dedekind) cut in F, so we may set $\mathbb{R}_F := \{A \subseteq \mathbb{Q}_F \colon \text{cut}\}$. Prove

$$\forall A \in \mathbb{R}_F \colon \ \sup(A) \text{ exists} \ \wedge \ A = \{a \in \mathbb{Q}_F \colon a < \sup(A)\}$$

Then prove that $\sup \colon \mathbb{R}_F \to F$ is a bijection.

Problem 2: Prove that the irrationals are dense in $\mathbb{R}$; that is, prove

$$\forall x, y \in \mathbb{R} \colon \ x < y \ \Rightarrow \ \Big(\exists z \in \mathbb{R} \smallsetminus \mathbb{Q} \colon \ x < z \wedge z < y \Big)$$

Problem 3: Prove that

$$\forall x, y \in \mathbb{R} \, \forall m \in \mathbb{N} \colon \ x^{m+1} - y^{m+1} = (x - y) \sum_{k=0}^{m} x^k y^{m-k}$$

and then use it to show that

$$\forall x, y \in \mathbb{R} \, \forall n \in \mathbb{N} \smallsetminus \{0\} \colon \ 0 < y \wedge y \leqslant x \ \Rightarrow \ n(x-y)y^{n-1} \leqslant x^n - y^n \leqslant n(x-y)x^{n-1}$$

Here $a \leqslant z \leqslant b$ means $a \leqslant z \wedge z \leqslant b$.

Problem 4: (RUDIN) PAGE 22, EX 6 (Construction of exponential function)

Problem 5: (RUDIN) PAGE 22, EX 7 (Construction of logarithm)

Problem 6: (RUDIN) PAGE 22, EX 9 (Lexicographic ordering of $\mathbb{C}$) Make sure to answer the last question as well.

Problem 7: Recall that the cardinality $|A|$ of a finite set A is the unique $n \in \mathbb{N}$ such that A is in a bijective correspondence with $[0, n) := \{k \in \mathbb{N} \colon k < n\}$. Prove that for any finite sets A and B:

(1) $B \subseteq A \ \Rightarrow \ |B| \leqslant |A|$
(2) $|A \cup B| \leqslant |A| + |B|$
(3) $|A \times B| = |A| \cdot |B|$

Problem 8: Construct explicit bijections between the stated intervals in $\mathbb{R}$:

(1) $f \colon [0,1) \to (0,1)$
(2) $g \colon [0,1] \to (0,1)$
(3) $h \colon [0,1] \to \mathbb{R}$

Hint: In (1) and (2), decompose the intervals into unions of half-open intervals.