HW#4: due Wed 2/8/2023

This exercise offers more practice of ordered fields, supremum and infimum, and then gives one problem on cardinality. Problems 3 and 6 have natural counterparts for subsets of $\mathbb{R}$.

Problem 1: Let F be a non-empty set and let $E := \mathcal{P}(F)$. In earlier HW we showed that the subset relation $\subseteq$ defines a (partial) order on E. Prove that every set $A \subseteq E$ (empty or not) admits $\sup(A)$ and $\inf(A)$ by showing:

$$\forall A \subseteq E\colon \quad A \neq \varnothing \Rightarrow \Big(\sup(A) = \bigcup A \ \wedge\ \inf(A) = \bigcap A\Big)$$

Prove also that $\sup(\varnothing) = \varnothing$ and $\inf(\varnothing) = F$.

Problem 2: Let $\mathbb{N}$ be naturals and abbreviate $\mathbb{N}' := \mathbb{N} \smallsetminus \{0\}$. For $m, n \in \mathbb{N}'$ define $m|n := (\exists k \in \mathbb{N}\colon n = m \cdot k)$. Prove that this is a partial order and then show that

(1) $\forall A \subseteq \mathbb{N}'\colon A \neq \varnothing \Rightarrow \inf(A)$ exists

(2) $\forall A \subseteq \mathbb{N}'\colon A \neq \varnothing \ \wedge\ A$ bounded $\Rightarrow \sup(A)$ exists

Hint: The infimum is actually the greatest common divisor and the supremum is the least common multiple of all numbers in A.

Problem 3: Prove that for all $A \subseteq \mathbb{N}$:

$$A \text{ is Dedekind infinite} \quad \Leftrightarrow \quad A \text{ is unbounded}$$

Recall that a set is Dedekind infinite if it supports an injection into but not onto itself.

Problem 4: Prove that for any non-empty $A, B \subseteq \mathbb{Q}$ admitting suprema, we have

$$A \subseteq B \ \Rightarrow\ \sup(A) \leqslant \sup(B)$$

Then show that under these conditions also $A \cup B$ admits a supremum and show

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$$

Here $\max\{a, b\}$ equals a if $b \leqslant a$ and equals b otherwise.

Problem 5: Given two sets $A, B \subseteq \mathbb{Q}$, denote

$$A + B := \{a + b\colon a \in A \ \wedge\ b \in B\}$$

Assuming that both A and B are non-empty and admit suprema, prove that so does the set $A + B$ and show

$$\sup(A + B) = \sup(A) + \sup(B)$$

Problem 6: (RUDIN) PAGE 22, EX 5

Problem 7: (RUDIN) PAGE 22, EX 8