HW#3: due Wed 2/1/2023

This exercise revisits some facts about naturals left unproven in class. Then we practice the constructions of integers and rationals from a system of naturals and prove some interesting facts about ordered fields.

Problem 1: Let $(\mathbb{N}, 0, S)$ be a system of naturals and recall that m^n is, for each $m, n \in \mathbb{N}$, the n-th power of m defined recursively so that

$$m^0 = 1 \wedge \forall n \in \mathbb{N}\colon\ m^{S(n)} = m \cdot m^n$$

Prove that, for each $m \neq 0$, we have:

(1) $\forall r, s \in \mathbb{N}\colon m^{r+s} = m^r \cdot m^s$,

(2) $\forall r, s \in \mathbb{N}\colon m^{r \cdot s} = (m^r)^s$.

Addition and multiplication are as defined in class. It is fine to use the properties of these without proof.

Problem 2: For the same setting as in the previous problem, prove that each non-empty $A \subseteq \mathbb{N}$ contains a smallest element, i.e.,

$$\forall A \in \mathcal{P}(\mathbb{N}) \smallsetminus \{\emptyset\}\, \exists n \in A\, \forall m \in A\colon n \leqslant m$$

In particular, the Axiom of Choice holds for collections of non-empty subsets of $\mathbb{N}$.

Hint: Noting that 0 lies “below” all elements of $\mathbb{N}$, and thus “below” all elements of A, construct the largest natural with the latter property.

Problem 3: Given a system of naturals $(\mathbb{N}, 0, S)$, the integers $\mathbb{Z}$ can be identified with the set of equivalence classes $[(m, n)]$ of pairs of naturals (i.e., elements of $\mathbb{N} \times \mathbb{N}$) under the equivalence relation

$$(m, n) \overset{+}{\sim} (m', n') := \ m + n' = n + m'$$

Now define the relation $\preceq$ on $\mathbb{Z}$ by

$$[(m, n)] \preceq [(m', n')] \quad := \quad m + n' \leqslant m' + n$$

where $\leqslant$ on the right is the ordering relation in $\mathbb{N}$. Prove that $\preceq$ is well defined in the sense of being independent on the representative (m, n) and (m', n'). Then show that $\preceq$ is a total order on $\mathbb{Z}$.

Problem 4: A natural way to think of the rationals is as the set $\mathbb{Q}$ of equivalence classes $[(p, q)]$ of pairs of integers from $\mathbb{Z} \times (\mathbb{Z} \smallsetminus \{0\})$ under the equivalence relation

$$(p, q) \overset{\cdot}{\sim} (p', q') \ := \ p \cdot q' = p' \cdot q.$$

We then define addition on $\mathbb{Q}$ by

$$[(p, q)] + [(p', q')] := \big[(p \cdot q' + q \cdot p', q \cdot q')\big]$$

Prove that this is well defined in the sense that the right-hand side is independent of the choice of the representatives (p, q) and (p', q'). Properties of addition and multiplication on $\mathbb{Z}$ can be used without apology.

Problem 5: Let $(F, +, 0, \cdot, 1, \leqslant)$ be an ordered field. Prove the following facts:

(1) $\forall a, b \in F \colon 0 \leqslant b \Leftrightarrow a \leqslant a + b$
(2) $\forall a, b \in F \colon a \leqslant b \;\Rightarrow\; -b \leqslant -a$
(3) $\forall a, b \in F \colon (0 < a \;\wedge\; a \leqslant b) \Rightarrow b^{-1} \leqslant a^{-1}$

Make sure to use only the operations postulated in the definition of an ordered field.

Problem 6: Let $(F, +, 0, \cdot, 1, \leqslant)$ be an ordered field. For each $a \in F$ define

$$|a| := \begin{cases} a & \text{if } 0 \leqslant a \\ -a & \text{if } a < 0 \end{cases}$$

Prove the following:

(1) $\forall a \in F \colon 0 \leqslant |a|$
(2) $\forall a \in F \colon -|a| \leqslant a \leqslant |a|$
(3) $\forall a, b \in F \colon |a + b| \leqslant |a| + |b|$
(4) $\forall a, b \in F \colon |a \cdot b| = |a| \cdot |b|$

Then use induction to generalize (3) to the form

$$\forall n \in \mathbb{N}_F \; \forall a_0, \dots, a_n \in F \colon \quad \left| \sum_{i=0}^{n} a_i \right| \leqslant \sum_{i=0}^{n} |a_i|$$

Give a formal definition of the (intuitive) symbol $\sum_{i=0}^{n} a_i$ as well. *Note*: The inequality in (3) is called the *triangle inequality*.

Problem 7: Do the following rationality tests:

(1) Prove that the iterated radical expression $\sqrt[3]{5 - \sqrt{3}}$ is not a rational number.
(2) Determine whether $\sqrt{3 + 2\sqrt{2}} - \sqrt{2}$ is rational or not.
(3) Find all rational roots of the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$.
