HW#2: due Wed 1/25/2023

This exercise practices natural numbers. We start with three problems proofs by induction (where it is fine to use the usual conventions for working with numbers without apology) and then proceed to problems practicing arithmetic of naturals and rationals and consequences of having a natural ordering relation on these.

Problem 1: Use induction to prove that

$$\forall n \geqslant 1\colon \quad \sum_{k=1}^{n} k^3 = \Bigl(\sum_{k=1}^{n} k\Bigr)^2$$

Problem 2: Use arguments/calculations to cast $\sum_{k=1}^{n} k^4$ as a polynomial in n. Then prove your formula by induction.

Problem 3: Consider a proposition depending on a natural number $n \geqslant 1$ defined as

$$P(n) := \Bigl(\sum_{k=1}^{n} k = \frac{1}{2}\bigl(n+\tfrac{1}{2}\bigr)^2\Bigr)$$

Prove that $\forall n \geqslant 1\colon P(n) \Rightarrow P(n+1)$. Then use basic facts about artithmetic of rationals to prove that $P(n)$ is FALSE for all $n \geqslant 1$. How come that induction does not work?

Problem 4: Write $\mathbb{Q}$ for the set of rationals and $\mathbb{N}$ for the set of naturals, both in the standard intuitive form (meaning: do not agonize over how these are realized in our set theory). Consider the set $\mathbb{J} \subseteq \mathbb{Q}$ of rationals of the form $n + \frac{m}{m+1}$, where $n, m \in \mathbb{N}$, i.e.,

$$\mathbb{J} := \bigl\{n + \tfrac{m}{m+1}\colon n, m \in \mathbb{N}\bigr\}$$

and let $S\colon \mathbb{J} \to \mathbb{J}$ be defined by

$$\forall n, m \in \mathbb{N}\colon \quad S\bigl(n + \tfrac{m}{m+1}\bigr) := n + \tfrac{m+1}{m+2}$$

Prove that $(\mathbb{J}, 0, S)$ satisfies Peano axioms P1-P4 but fails P5. *Hint* : Find $\operatorname{Ran}(S)$. A follow-up question: Can you find $\mathbb{K} \subseteq \mathbb{J}$ such that $(\mathbb{K}, 0, S)$ *is* a system of naturals?

Problem 5: Recall that we defined $m + n$ for $m, n \in \mathbb{N}$ recursively so that

$$\forall m \in \mathbb{N}\colon \quad m + 0 = m \wedge \Bigl(\forall n \in \mathbb{N}\colon\ m + S(n) = S(m+n)\Bigr)$$

where 0 is the zero element and S is the successor function. Prove that addition on $\mathbb{N}$ is associative, i.e.,

$$\forall k, n, m \in \mathbb{N}\colon \quad n + (m+k) = (n+m) + k$$

Do not assume that addition is commutative as that requires a proof that is considerably longer than what you are asked to produce.

Problem 6: Write the number commonly known as "ten thousand" in base-6 system. (You should be able to do this without relying on a calculator.)

Problem 7: Recall that the ordering relation $\leqslant$ on $\mathbb{N}$ is defined by

$$m \leqslant n \;:=\; \exists k \in \mathbb{N}\colon\, n = m + k$$

Prove that

(1) $\forall n \in \mathbb{N}\colon 0 \leqslant n$
(2) $\forall n \in \mathbb{N}\colon n \leqslant S(n)$
(3) $\forall m, n \in \mathbb{N}\colon m \leqslant n \Rightarrow S(m) \leqslant S(n)$

Writing $m \cdot n$ for multiplication on $\mathbb{N}$ which is defined recursively by

$$\forall m \in \mathbb{N}\colon \quad 0 \cdot m = 0 \wedge \Big(\forall n \in \mathbb{N}\colon S(n) \cdot m = n \cdot m + m\Big)$$

prove also

$$\forall m, n, r \in \mathbb{N}\colon\;\; m \leqslant n \Rightarrow r \cdot m \leqslant r \cdot n$$

Write down all the facts about addition, multiplication and $\leqslant$ you are using.

Problem 8: Denote $[0, n) := \{k \in \mathbb{N}\colon k < n\}$. Let $m, n \in \mathbb{N}$ and assume $h\colon [0, n) \to [0, m)$ is a function. Prove the following:

(1) h injective $\;\Rightarrow\;$ $n \leqslant m$
(2) h surjective $\;\Rightarrow\;$ $m \leqslant n$
(3) h bijective $\;\Rightarrow\;$ $n = m$

Remember that (for us) being injective means that the function is everywhere defined. Part (3) shows that every finite set has unique cardinality.
