HW#1: due Wed 1/18/2023

This exercise practices the formalism of propositional logic and operations on sets which we will need throughout the course.

Problem 1: Let P and Q be logical propositions (i.e., statements with TRUE/FALSE value). Write the truth tables of the propositions

(1) $(P \vee Q) \wedge \neg(P \wedge Q)$
(2) $P \Rightarrow (Q \Rightarrow \neg P)$

Then use the truth table to check that

$$P \Leftrightarrow \Big((P \wedge \neg Q) \vee (P \wedge Q)\Big)$$

is TRUE regardless of the truth values of P and Q, and is thus a *tautology*.

Problem 2: Assume the intuitive form of the naturals $\mathbb{N}$ endowed with the operation of multiplication. Write the following verbal statements using predicate calculus (i.e., logical propositions, predicates and quantifiers): First define the proposition $m|n$ meaning "n is divisible by m." Then use it to write the statements:

(1) every n that is divisible by 3 is divisible by 7 only if it is even,
(2) some but not all n that are divisible by 6 are divisible by 4,
(3) not every n that is divisible by 6 is divisible by 5 but 20 divides those that are,
(4) logical negation of (1)
(5) logical negation of (2)
(6) logical negation of (3)

The conversion from idiomatic English to propositional logic is not a mathematical operation and so you may need to interpret the statement properly first. Also note that some of these statements may not be TRUE but, since FALSE propositions are still propositions, this is irrelevant for the task at hand.

Problem 3: Working with real numbers this time, write the following statements in the formal language of propositional logic:

(1) For each set A of reals, there exists exactly one $x \in A$ such that $x^2 = 1$.
(2) For each set A of reals all but one $x \in A$ is smaller than x.
(3) No real x belongs to every non-empty set A of reals.

Again, it is immaterial for this task whether this statement is TRUE or FALSE.

Problem 4: Let C be a set and consider the relation

$$A \subseteq B := (\forall x \in A\colon x \in B)$$

on the powerset $\mathcal{P}(C)$ of C. Prove that this relation is reflexive, antisymmetric and transitive (and is thus a partial order).

Problem 5: Let $\{A_\alpha \colon \alpha \in I\}$ be a non-empty collection of subsets of Y. For $A \subseteq Y$, let $A^{\mathrm{c}} := Y \setminus A$. Prove de Morgan's laws:

$$\bigcup_{\alpha\in I} A_\alpha^{\mathrm{c}} = \Big(\bigcap_{\alpha\in I} A_\alpha\Big)^{\mathrm{c}}$$

and

$$\bigcap_{\alpha\in I} A_\alpha^{\mathrm{c}} = \Big(\bigcup_{\alpha\in I} A_\alpha\Big)^{\mathrm{c}}$$

Remember that (by Axiom of Extensionality) two sets are equal if and only if they have the same elements.

Problem 6: Let $\sim$ be an equivalence relation on a set A and recall that the equivalence class of $x \in A$ is defined as

$$[x] := \{y \in A \colon x \sim y\}$$

Prove that two equivalence classes are either disjoint or equal, i.e.,

$$\forall x, y \in A \colon [x] = [y] \,\vee\, [x] \cap [y] = \varnothing$$

Problem 7: Given two sets A and B, recall that $A \times B$ is a set of ordered pairs (x, y) that are themselves defined as $(x, y) := \{\{x\}, \{x, y\}\}$. Prove that

$$\forall x, \tilde{x} \in A\, \forall y, \tilde{y} \in B \colon \quad (x, y) = (\tilde{x}, \tilde{y}) \,\Leftrightarrow\, x = \tilde{x} \,\wedge\, y = \tilde{y}$$

Note: $\{\{x\}, \{x, y\}\}$ is called the Kuratowski pair.

Problem 8: Let $f \colon X \to Y$ be a function and $\{Y_\alpha \colon \alpha \in I\}$ a collection of subsets of Y. Recall that the preimage of set $B \subseteq Y$ under f is the set $f^{-1}(B) := \{x \in X \colon f(x) \in B\}$. Prove the following equalities:

$$f^{-1}\Bigg(\bigcup_{\alpha\in I} Y_\alpha\Bigg) = \bigcup_{\alpha\in I} f^{-1}(Y_\alpha)$$

and

$$f^{-1}\Bigg(\bigcap_{\alpha\in I} Y_\alpha\Bigg) = \bigcap_{\alpha\in I} f^{-1}(Y_\alpha)$$
