HW#3: due Fri 10/25/2019
(This assignment comes on two pages.)

Problem 1: Ex 3.8, PAGE 19

Problem 2: Ex 4.12, PAGE 27

Problem 3: Ex 4.14, PAGE 28

Problem 4: Prove that for any $A, B \subseteq \mathbb{R}$,
$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$$

Problem 5: Let $E$ be a set and $\mathcal{P}(E)$ be the powerset of $E$ — namely, the set of all subsets of $E$. The relation $A \subseteq B$ defines a (partial) order on $\mathcal{P}(E)$. Prove that every nonempty set $F \subseteq \mathcal{P}(E)$ admits sup$(F)$ and inf$(F)$ by showing:
$$\forall F \subseteq \mathcal{P}(E): \ F \neq \emptyset \Rightarrow \sup(F) = \bigcup F \wedge \inf(F) = \bigcap F$$

Now take $F$ to be set of the elements in a sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets of $E$. Abbreviating $\sup_{m \geq n} A_m := \sup\{A_m: m \geq n\}$ and $\inf_{m \geq n} A_m := \inf\{A_m: m \geq n\}$, define the limes superior and limes inferior by
$$\limsup_{n \to \infty} A_n := \inf_{m \geq 0} \sup_{n \geq m} A_n$$
$$\liminf_{n \to \infty} A_n := \sup_{m \geq 0} \inf_{n \geq m} A_n$$

Prove that
$$\liminf_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n$$

If equality is TRUE, then we say that the limit of $A_n$ exists and define $\lim_{n \to \infty} A_n$ by the common value of limsup and liminf. Assuming this to be the case, prove
$$\lim_{n \to \infty} A_n = \{x \in E: (\exists n \in \mathbb{N}\ \forall m \in \mathbb{N}: m \geq n \Rightarrow x \in A_m)\}$$

NOTE: Throughout this problem, we are talking about sequences of sets and various limit notions thereof. This may appear to be very different from limits of numbers but it is not, once we realize real numbers as subsets of the rationals called Dedekind cuts.

Problem 6: Ex 7.5, PAGE 38

Problem 7: Ex 8.2(C), PAGE 44
Problem 8: EX 8.2(E), PAGE 44

Problem 9: EX 8.6, PAGE 44

Problem 10: Determine if
\[ \lim_{n \to \infty} \left( \sqrt[3]{n^3 + n^2 + 1} - \sqrt[3]{n^3 + 1} \right) \]
exists or not. Prove your claim.

Problem 11: Define \( a_n \) recursively by
\[ a_0 := 1 \land \left( \forall n \in \mathbb{N} : a_{n+1} = \sqrt{2 + a_n} \right) \]
Determine if \( \lim_{n \to \infty} a_n \) exists and if so, find its value. Prove your claim.