LEVEL SET BASED MULTISPECTRAL SEGMENTATION WITH CORNERS *

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Abstract. In this paper we propose an active contour model for segmentation based on the Chan-Vese model. The new model can capture inherent sharp features, i.e., the sharp corners of objects, which are often smoothed by the regularization term in segmentation. Motivated by the snaked based method in (Droske and Bertozzi SIAM J. on Image Sci. 2010) that emphasizes straight edges and corners without regard to orientation, we develop a region-based method with a level set representation. The model combines the Chan-Vese model with the level set version of a higher order nonlinear term. We extend this model to multispectral images. Higher order methods can be very stiff, so we propose a splitting scheme to remove the stiffness and prove its stability and convergence. Finally we show numerical results on gray, color and hyperspectral images. We can see that the model is robust to noise.

Key words. segmentation, corners, high order and nonlinear, level set representation, numerical stability and convergence

AMS subject classifications. 35G20, 65M06, 65M12

1. Introduction. Segmentation is one of the most important tasks in image processing. The main idea of image segmentation is to detect the objects in the given image. Usually, this is done by evolving a curve towards the boundary of the object. Generally speaking, the existing segmentation methods can be divided into two categories: curve based methods and region based methods. The curve based methods include the 'snake' model by Kass et al [20] and geodesic active contour model by Caselles et al [8]. The region based methods include Mumford-Shah [23] and related Chan-Vese [12] methods. We briefly describe these methods.

Kass et al (1988) [20] developed the 'snake' or active contour model. In the snake model, the curve evolution is obtained by minimizing a carefully designed functional energy. Let Ω be a bounded and open subset of $\mathbb{R}^2$, with $\partial\Omega$ its boundary. Let f be the given image, as a bounded function defined on $\overline{\Omega}$ with real values. Usually $\overline{\Omega}$ is a rectangular domain. Let $C(q) : [0, 1] \to \mathbb{R}^2$ be a parametrized curve. Then the snake method is minimizing the following functional energy:

$$E(C) = \alpha \int_0^1 |C'(q)|^2 dq + \beta \int_0^1 |C''(q)|^2 dq - \lambda \int_0^1 |\nabla f(C(q))|^2 dq. \qquad (1.1)$$

The first two terms, the membrane energy and the elasticity energy, control the smoothness of the curve. They are called the internal energy. The third term is the external term and depends on the image data. It is easy to see that the external energy term is small when the gradient of f has a large magnitude, thus pushing the curve towards edges. Such functions are usually called edge detectors. The active contour model was further developed by [8, 6, 7, 21, 28] using different edge detectors. For example, Caselles et al [8] introduced a geodesic active contour model by

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minimizing the functional energy

$$E_g(C) = \alpha \int_0^1 |C'(q)|^2 dq + \lambda \int_0^1 g(|\nabla f(C(q))|)^2 dq \tag{1.2}$$

where $g: \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function such that $\lim_{x\to\infty} g(x) = 0$. In the geodesic active contour model, $-|\nabla f|^2$ is replaced by $g(|\nabla f|)^2$. In addition, considering that the snake model does not allow topology change in the curve evolution, and consequently can only detect one object in the image, Caselles et al [8] employed a level set representation building on the pioneering work of Osher and Sethian [24]. Let $\phi(t, \cdot)$ be a level set function such that $C(q)$ is the zero level set of ϕ, i.e., $C(q,t) = \{x \in \mathbb{R}^2 : \phi(t,x) = 0\}$, then the level set function can be evolved instead of the curve. Caselles proposed the following evolution equation for ϕ

$$\phi_t = |\nabla\phi| \mathrm{div}\big(g(f)\frac{\nabla\phi}{|\nabla\phi|}\big) + cg(f)|\nabla\phi| = g(c+\kappa)|\nabla\phi| + \nabla\phi\cdot\nabla g,$$

where κ is the curvature and c is a constant.

Mumford and Shah [23] addressed an active contour model by minimizing the following energy

$$E_{MS}(u, C) = \mu \cdot \mathrm{length}(C) + \lambda \int_{\Omega\backslash C} |u-f|^2 dx + \int_{\Omega\backslash C} |\nabla u|^2 dx, \tag{1.3}$$

where the boundary curve C is exactly the set of discontinuity of u. Morel and Solimini [22] considered the special case that u is piecewise constant, thus the gradient term ∇u is zero on $\Omega\backslash C$ and the model becomes

$$E(u, C) = \mu \cdot \mathrm{length}(C) + \lambda \int_{\Omega\backslash C} |u-f|^2 dx. \tag{1.4}$$

This model addressed the simplest balance between accuracy of the regions and parsimony of the boundaries. Morel and Solimini [22] also presented some computational theories for the piecewise constant model.

Chan and Vese [12] formulated a binary case variant of the piecewise constant model, and the boundary curve C was represented by a level set function ϕ satisfying $\phi > 0$ inside C and $\phi < 0$ outside C. By defining the Heaviside function $H(x) = 1_{x\geqslant 0}$ and the one-dimensional Dirac measure $\delta = \frac{d}{dx}H(x)$, the functional energy became

$$\begin{aligned} E_{CV}(\phi, c_1, c_2) &= \mu \int_\Omega \delta(\phi)|\nabla\phi| + \nu \int_\Omega H(\phi) dxdy \\ &+\lambda_1 \int_\Omega (f-c_1)^2 H(\phi) dxdy + \lambda_2 \int_\Omega (f-c_2)^2 (1-H(\phi)) dxdy. \end{aligned} \tag{1.5}$$

The gradient descent equation for Chan and Vese active contour model is

$$c_1 = \frac{\int_\Omega fH(\phi)dxdy}{\int_\Omega H(\phi)dxdy}, \qquad c_2 = \frac{\int_\Omega f(1-H(\phi))dxdy}{\int_\Omega (1-H(\phi))dxdy},$$

$$\phi_t = \delta(\phi)\big[\mu\nabla\cdot\frac{\nabla\phi}{|\nabla\phi|} - \nu - \lambda_1(f-c_1)^2 + \lambda_2(f-c_2)^2\big]. \tag{1.6}$$

with boundary condition $\frac{\delta(\phi)}{|\nabla\phi|}\frac{\partial\phi}{\partial\vec{n}} = 0$. This model was further extended in [11, 10]. Moreover, there are fast algorithms for solving the Chan-Vese model, including the methods by Chambolle [9] and Pan et al [25].

All the segmentation models above minimize a functional energy including an edge detecting term and a regularization term which is usually the length of the curve. As is known, the regularization terms avoid local minima and ensure the smoothness of the boundary curve, especially when the image is noisy. However, they often introduce undesired over-smoothing to the sharp features, especially corners. If the complete information about the morphology and anisotropy in the image is known, for example, the orientation of buildings in an aerial photograph, then it is natural to minimize some anisotropic functional to obtain segmentation with corners. This idea comes from the study of the Wulff-shapes in material science. Numerical methods have been developed for anisotropic flows in [1, 5, 14, 15, 16, 18, 19]. However, in a typical application the orientation and morphology are not known, a prior must be inferred from the data. Consequently we focus on automatic detection guided by the intrinsic geometric features in the image.

Droske and Bertozzi [17] proposed a new algorithm based on the snake method and motivated by the low curvature image simplifier (LCIS), which is known for preserving jump discontinuity in slope. By combining the geodesic snake construction with nonlinear diffusion of edges, they are successful in segmenting objects with sharp corners. However, the method still suffers from other common drawbacks of snake-based methods. In particular, one can not naturally perform topology changes and moreover, a multi scale preprocess of the image is required to avoid local minima due to clusters in the image. This prompts us to develop another segmentation model.

This paper is organized as follows. In the next section we review the work of Droske and Bertozzi [17] discuss the properties of the high order equations. Then we formulate the corner preserving term in a level set framework. By combining the corner preserving term with the Chan-Vese model, we obtain a new model that inherit the merits of Chan-Vese model as well as one that retains the sharp corners. In addition, we extend this model to the color and hyperspectral images. In section 3 we describe the numerical implementation details of the high order nonlinear PDE. We also prove the convergence of the time stepping scheme. In section 4 we validate our model by numerical tests on gray, color and hyperspectral images, and we end the paper by a brief conclusion section.

2. Chan-Vese with corner preserving term. The new method developed in [17] is motivated by the $low-curvature\ image\ simplifier$ (LCIS), which is first introduced by Tumblin and Turk [32] and later developed by Bertozzi and Greer [3]. As demonstrated in the numerical examples in [32, 3] for one dimensional signal denoising problems, the isotropic diffusion will impose oversmoothing on the corners, while the anisotropic diffusion will generate staircases. Thus neither can handle continuous signals with some corner-shape transitions. To overcome the drawbacks of the existing models, Tumblin and Turk proposed the following fourth-order equation

$$u_t + \text{div}(g(\Delta u)\nabla\Delta u) = 0. \tag{2.1}$$

Here g is typically a weight function, with $g(0) = 1$ and $g(s) \to \infty$ as $s \to \infty$. In [32, 3], they choose $g(s) = \left(1 + \frac{s^2}{\eta^2}\right)^{-1}$ by analogy with the Perona-Malik method in [26], where η is a positive parameter.

The high order equation (2.1) imposes stronger smoothness requirement than the isotropic and anisotropic diffusion models, thus eliminates the stair cases. In addition, Bertozzi and Greer [3] showed that for one dimensional signal denoising problems, equation (2.1) could be combined with an L^2 fidelity term to generate "piecewise linear" solutions. The solution u is actually a smooth function and the corners are understood in an infinitesimal sense. In [3] the authors gradually decreased the grid size to demonstrate that the transitions are actually smooth if the resolution is high enough. Although this can be shown for one dimensional signals by using many grid points, the same resolution in two dimensions would be prohibitively expensive for out calculation. Moreover, this would require ultra high resolution data which we do not have.

Furthermore, equation (2.1) is a gradient flow of the non-quadratic energy functional $E_G(u) = \int_\Omega G(\Delta u)dx$, where G is the antiderivative of g. It also decreases the H^1 energy $E(u) = \int_\Omega |\nabla u|^2 dx$. Bertozzi and Greer [3] proved the existence of global smooth solutions in the one dimension case with the same choice of g as in [26]. Nevertheless, the existence of global solutions in higher dimensions remains an open problem.

Motivated by the corner preserving property and the fact that the contour curve is actually one-dimensional, Droske and Bertozzi [17] addressed a modified snake method. By replacing the gradient and Laplace operators by the corresponding intrinsic surface gradient and surface Laplace operators, and by choosing the coordinate x as the free variable, they obtain a straightforward variant of equation (2.1) as follows:

$$x_t - \mathrm{div}_\Gamma(g(h)\nabla_\Gamma \Delta_\Gamma x) = 0. \tag{2.2}$$

where ∇_Γ, div_Γ, Δ_Γ are the intrinsic surface gradient, surface divergence and surface Laplace operators respectively. The authors combined this equation with some classical snake methods and obtained favorable numerical results.

Let $\vec{n}$ be the outer normal vector of the surface Γ and h be the mean curvature, then we have an interesting equality: $\Delta_\Gamma x = h\vec{n}$. By plugging this equality in equation (2.2) and keeping the dominant fourth order term, we obtain the following equation.

$$x_t - \mathrm{div}_\Gamma(g(h)\nabla_\Gamma h)\vec{n} = 0. \tag{2.3}$$

Equation (2.3) is a surface evolution equation with velocity function $g(s)$ that only depends on the mean curvature h. Especially, if we take $g(s) \equiv 1$, we obtain the evolution equation of motion by surface Laplacian of mean curvature, which has been discussed numerically in [27, 30] via level set formulation. Droske and Bertozzi also pointed out that equation (2.3) behaves quite similarly to equation (2.2). Furthermore, equation (2.3) can preserve the area enclosed by Γ and decrease the length of Γ like the regular surface diffusion. Therefore, we can use (2.3) to replace the length regularization term in active contour models, although [17] uses (2.2) for simpler numerical implementation.

The main purpose of this manuscript is to recast this equation in terms of a level set formulation, and to illustrate its usefulness in segmenting complex images with sharp corners. Following the level set representation of the geometric features in Chopp et al [13] and Bertamio et al [2], we obtain the level set version of equation (2.3), which is fourth order and nonlinear. While the derivation of the equation is straightforward, the main challenge in numerical implementation is to develop an efficient time stepping scheme. For example, explicit schemes usually require that $dt \sim dx^4$, which is very restrictive. In section 3 we propose an efficient splitting scheme and prove its convergence.

Suppose the initial surface is given by the zero level set of a function $\phi(\cdot, 0)$, or, $\Gamma(0) = \{x : \phi(x, 0) = 0\}$, and the surface at time t is the zeros level set of $\phi(\cdot, t)$. The normal direction is given by $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$ and the mean curvature is given by $h = \mathrm{div}(\vec{n})$ for any point on the curve Γ. Further we need to define the intrinsic surface gradient, surface divergence and surface Laplacian operators via level set representation. In [2], Bertalmio et al derived all these operators via level set representation and solved the PDE on surfaces. According to their work, the surface gradient is simply the projection of the gradient operator onto the tangent plane:

$$\nabla_\Gamma \phi = \nabla\phi - (\nabla\phi \cdot \vec{n})\vec{n}.$$

The surface divergence operator div_Γ is the dual operator of the surface gradient operator, and the surface Laplacian, or the Laplace-Beltrami operator is given by

$$\Delta_\Gamma \phi = \mathrm{div}_\Gamma(\nabla_\Gamma \phi).$$

Now we only need to convert the corner preserving equation into a level set formulation. We are more interested in equation (2.3) than (2.2), because of the length decreasing property and simpler numerical implementation. With the level set representation above, the level set version of equation (2.3) can be written as:

$$\phi_t = -|\nabla\phi| \mathrm{div}_\Gamma(g(h)\nabla_\Gamma h). \tag{2.4}$$

To demonstrate the curve evolution by equation (2.4), we repeat the following numerical curve evolution test in Droske et al [17]. The initial curve is chosen in polar coordinates as $r = \frac{1}{2} + \frac{1}{10}\sin(15\theta)$, where r and θ are just the classic polar coordinate parameters: $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{x}$. The initial level set function is $u = r - \frac{1}{2} - \frac{1}{10}\sin(15\theta)$. The curve evolution is shown in Figure (2.1). Using the level set representation, the black curve is the zeros level set of the function ϕ and the color in the figure stands for the value of the level set function. During the evolution, the initial smooth curve develops corners quickly by accentuating the high curvature parts. The corners keep existing until the curve eventually converges to a circle by the infinitesimal regularity.

The idea of this manuscript comes from the process above: if the curve evolution is combined with a fidelity term, we can expect the curve to stop at a stable state with sharp corners. This prompts us to combine the Chan-Vese model with the equation (2.4). With the fitting term in Chan-Vese model, we get the following equation.

$$\begin{aligned} \phi_t = &-\alpha|\nabla\phi| \mathrm{div}_\Gamma(g(h)\nabla_\Gamma h) \\ &+ \delta(\phi)\big[\mu\nabla \cdot \frac{\nabla\phi}{|\nabla\phi|} - \nu - \lambda_1(f - c_1)^2 + \lambda_2(f - c_2)^2\big]. \end{aligned} \tag{2.5}$$

For multiband images, let N be the number of bands and f_i be the gray value of the ith band. Using the technique in [28, 11], we can similarly calculate the c_{1i} and c_{2i} of the ith band with f_i and the level set function ϕ, and then we obtain the level set evolution equation for multi-band images by simply taking the algebraic average of the gradient descent flow for each band:

$$\begin{aligned} \phi_t = &-\alpha|\nabla\phi| \mathrm{div}_\Gamma(g(h)\nabla_\Gamma h) \\ &+ \delta(\phi)\big[\mu\nabla \cdot \frac{\nabla\phi}{|\nabla\phi|} - \nu - \frac{1}{N}\sum_{i=1}^{N}\lambda_{1i}(f_i - c_{1i})^2 + \frac{1}{N}\sum_{i=1}^{N}\lambda_{2i}(f_i - c_{2i})^2\big]. \end{aligned} \tag{2.6}$$

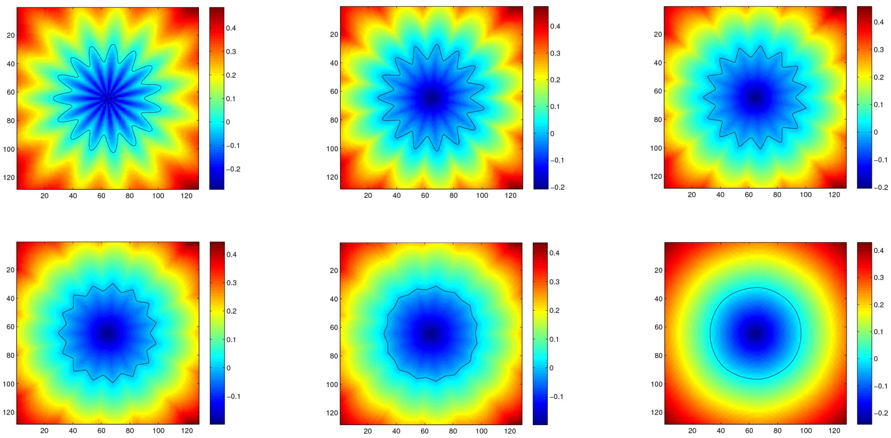


FIG. 2.1. *The evolution of a curve. We can see that corners are formed in early stage.*

This can also be combined with the segmentation method with spectral angle by Ye [33], in which the authors used spectral angle for hyperspectral images instead of the Chan-Vese fidelity term.

As we mentioned above, the corner preserving term can decrease the curve length and impose regularization on the level set function. Therefore, we can drop the length term in the Chan-Vese model and only use the corner preserving term. Solving high order nonlinear equations is usually difficult, since the stability condition is more restrictive. We will describe the numerical scheme in the next section.

3. Semi-Implicit Numerical Scheme.

Equation (2.5) and (2.4) are fourth order nonlinear equations. For the numerical implementation, if we apply an explicit numerical scheme, the nonlinear high order equation usually requires a time step $dt \sim dx^4$, which leads to very slow evolution. If we attempt a fully implicit numerical scheme, then solving the nonlinear implicit equation at each time step is difficult. As a result, semi-implicit schemes are preferred for this kind of equation. We consider the numerical scheme introduced by Smereka, Salac and Lu [27, 30] for the curve evolution by surface Laplacian of mean curvature. Although this method has been discovered and implemented numerically in the literature in these papers, a rigorous proof of convergence remains new. We extend the scheme in [27, 30] to the more general cases studied here, and prove convergence of the time stepping scheme.

For simplicity, we write the equation (2.5) as $\phi_t = S(\phi)$. We add a bilaplacian stabilization term to both sides of the PDE and obtain

$$\phi_t + \beta\Delta^2\phi = S(\phi) + \beta\Delta^2\phi. \tag{3.1}$$

with β a positive constant. To distinguish the exact solution from the numerical solution, we use upper case and bold characters for the numerical solution, lower case for the exact solution. In other words, we write Φ^k, $\mathbf{h}^k$, $\nabla_{\mathbf{\Gamma}}$ and $\Delta_{\mathbf{\Gamma}}$ for the numerical equation at the kth step, while ϕ^k, h^k, ∇_Γ and Δ_Γ for the exact solution at time $k \cdot dt$. Let $e^k = \phi^k - \Phi^k$ denote the discretization error. Taking the left side bilaplacian at the new time level and the entire right side at the old time level, we obtain

$$\frac{\Phi^{k+1} - \Phi^k}{dt} + \beta\Delta^2\Phi^{k+1} = \beta\Delta^2\Phi^k + S(\Phi^k), \tag{3.2}$$

which is equivalent to

$$(\Phi^{k+1} - \Phi^k) = dt(1 + dt \cdot \beta\Delta^2)^{-1} S(\Phi^k). \tag{3.3}$$

Here the operator $(1 + dt \cdot \beta\Delta^2)^{-1}$ is positive definite, thus it works as a smoothing operator. Empirically we choose the parameter $\beta = 1/2$ as in [27, 30]. The equation can be solved with Fast Fourier Transform (FFT). As shown in [27, 30], the numerical experiments suggests that we can take $dt \sim dx^2$. This is a great improvement compared to $dt \sim dx^4$ for explicit schemes.

For image processing problems, we usually take the domain $\Omega = [0, 1) \times [0, 1)$. In the following part we outline the discretization of equation (2.5). The right hand side is composed of two parts, the Chan-Vese energy term and the corner preserving term. For the Chan-Vese energy term, we simply follow the numerical discretization in [12]. We focus on the corner preserving term.

With the level set representation, the outer normal direction $\vec{\mathbf{n}}$ is $\vec{\mathbf{n}} = (\mathbf{n}^x, \mathbf{n}^y) = \frac{\nabla\Phi}{|\nabla\Phi|}$, and the mean curvature can be represented as $\mathbf{h} = \mathrm{div}(\vec{\mathbf{n}})$. However, in the actual implementation, we usually use $|\nabla\Phi|_\delta = (\Phi_x^2 + \Phi_y^2 + \delta^2)^{1/2}$ instead of $|\nabla\Phi| = (\Phi_x^2 + \Phi_y^2)^{1/2}$ to avoid division by zero, where δ is a small parameter. Consequently the modified normal direction and mean curvature are

$$\vec{\mathbf{n}}_\delta = (\mathbf{n}_\delta^x, \mathbf{n}_\delta^y) = \frac{\nabla\Phi}{|\nabla\Phi|_\delta} = \left(\frac{\Phi_x}{(\Phi_x^2 + \Phi_y^2 + \delta^2)^{1/2}} \, , \, \frac{\Phi_y}{(\Phi_x^2 + \Phi_y^2 + \delta^2)^{1/2}} \right), \tag{3.4}$$

and

$$\begin{aligned} \mathbf{h}_\delta = \mathrm{div}(\vec{\mathbf{n}}_\delta) &= \frac{\Delta\Phi}{|\nabla\Phi|_\delta} - \frac{\nabla\Phi^T \nabla^2\Phi\nabla\Phi}{|\nabla\Phi|_\delta^3} \\ &= \frac{\Phi_{xx} + \Phi_{yy}}{(\Phi_x^2 + \Phi_y^2 + \delta^2)^{1/2}} - \frac{\Phi_x^2\Phi_{xx} + 2\Phi_x\Phi_y\Phi_{xy} + \Phi_y^2\Phi_{yy}}{(\Phi_x^2 + \Phi_y^2 + \delta^2)^{3/2}}. \end{aligned} \tag{3.5}$$

For numerical analysis, we make the same modification for the original equations (2.4) and (2.5), i.e., we use the modified $|\nabla\phi|_\delta = (\phi_x^2 + \phi_y^2 + \delta^2)^{1/2}$ instead of $|\nabla\phi| = (\phi_x^2 + \phi_y^2)^{1/2}$, and consequently use modified $\vec{n}_\delta$ and h_δ instead of $\vec{n}$ and h. As long as the parameter δ is small enough, the zero level set of the modified equation is a good approximation to that of the original equation. In addition, the modified equation can avoid singularities at the local maxima and minima of the level set function ϕ. Therefore, in the following analysis we always discuss the modified equations (2.4) and (2.5). Since the main difficulty for numerical implementation is the surface Laplacian term, which is fourth order and nonlinear, we focus on the equation (2.4) rather than equation (2.5). The modified equation and corresponding numerical scheme goes as follows.

$$\phi_t = -|\nabla\phi|_\delta \mathrm{div}_\Gamma(g(h_\delta)\nabla_\Gamma h_\delta), \tag{3.6}$$

$$\frac{\Phi^{k+1} - \Phi^k}{dt} + \beta\Delta^2\Phi^{k+1} = \beta\Delta^2\Phi^k - |\nabla\Phi^k|_\delta \mathrm{div}_{\mathbf{\Gamma}}\big(g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k\big). \tag{3.7}$$

To compute $\mathrm{div}_{\mathbf{\Gamma}}(g(\mathbf{h}_\delta)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta)$, we may take the surface gradient of the mean curvature $\mathbf{h}_\delta$, and then calculate the surface divergence of $g(\mathbf{h}_\delta)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta$. However, we prefer to calculate the surface Laplacian of $G(\mathbf{h})$ as in [3], where G is the antiderivative of g, or, $G'(s) = g(s)$. In our numerical method, we choose $g(s) = \left(1 + \frac{s^2}{\eta^2}\right)^{-1}$ and $G(s) = \frac{1}{\eta} \arctan\left(\frac{s}{\eta}\right)$ where η is a positive parameter. According to the definition of surface gradient $\nabla_{\mathbf{\Gamma}} G = \nabla G - (\nabla G \cdot \vec{\mathbf{n}_\delta})\vec{\mathbf{n}_\delta}$, we have the following component form $(\nabla G \cdot \vec{\mathbf{n}_\delta}) = \mathbf{n}_\delta^x G_x + \mathbf{n}_\delta^y G_y$, where the subscripts on G denotes the partial derivatives in x and y individually. Therefore we can write the surface gradient as

$$\begin{aligned} \nabla_{\mathbf{\Gamma}} G &= G_x e^x + G_y e^y - (\mathbf{n}_\delta^x G_x + \mathbf{n}_\delta^y G_y)(\mathbf{n}_\delta^x e^x + \mathbf{n}_\delta^y e^y) \\ &\equiv A e^x + B e^y, \end{aligned} \tag{3.8}$$

where e^x and e^y are unit vectors in the x and y direction respectively. By computing the surface divergence in a similar way we can obtain the surface Laplacian of $G(\mathbf{h}_\delta)$

$$\Delta_{\mathbf{\Gamma}} G = A_x + B_y - \mathbf{n}_\delta^x(\mathbf{n}_\delta^x A_x + \mathbf{n}_\delta^y A_y) - \mathbf{n}_\delta^y(\mathbf{n}_\delta^x B_x + \mathbf{n}_\delta^y B_y). \tag{3.9}$$

Next we will analyze this semi-implicit scheme with some more details and rigorous estimates for the numerical solution. We use similar technique as in Bertozzi et al [4] and Schoenlieb et al [29], focusing on discretization in time. Denote $|D^m u|^2 = \sum_{|\alpha|=m} |\partial^\alpha u|^2$ and $\|D^m u\|^2 = \sum_{|\alpha|=m} \|\partial^\alpha u\|^2$ for any integer m, where $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $\partial^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$. Due to the high order and nonlinearity, we need several restrictions on the smoothness of the level set function. The results are summarized in the following theorem.

THEOREM 3.1. *Let ϕ be the exact solution of (3.6) and $\phi^k = \phi(kdt)$ be the exact solution at time kdt for a time step $dt > 0$ and $k \in \mathbb{N}$. Let Φ^k be the kth iterate of (3.7). Assume that there exits a constant L such that $|g(s)| \leqslant L$, $|g'(s)| \leqslant L$, and the discrete solution exists up to time T, then we have the following statements:*

(i) Under the assumption that $\|\phi_{tt}\|_{-1}$, $\|\nabla\Delta\phi_t\|_2$, $\|\nabla\phi\|_\infty$ and $\|\phi_t\|_{-1}$ are bounded, the numerical scheme (3.7) is consistent with the modified continuous equation (3.6) and first order in time.

(ii) Let further $e^k = \phi^k - \Phi^k$ be the discretization error. If

$$\|\partial^\alpha \phi^k\|_\infty \leqslant K, \qquad \|\nabla \Phi^k\|_\infty \leqslant K,$$

for a constant $K > 0$ and all $|\alpha| \leqslant 3$, $kdt \leqslant T$, then the error e^k converges to zero with first order in time.

REMARK: 1. Although the following convergence proof only requires dt smaller than some constant which is independent of dx, the assumption that the derivatives of ϕ are bounded may impose additional restriction on the time step dt. In fact, for the most commonly used level set function, the signed distance function, $|\nabla\phi|$ is usually unbounded. In addition, all the constants depend on the choice of δ. However, we have to take δ small to make sure that the solution of the modified equation is close to the solution of the original equation. We may have to take dt small enough to obtain desired accuracy.

REMARK: 2. Solving the equation in a narrow band of the zero level set may reduce the singularity of the level set function. For example, the signed distance function is singular in the whole domain, but it is smooth in a small neighborhood of the zero level set, as long as the zero level curve is smooth. In addition, in the numerical implementation, we impose an upper bound K for $|\nabla\Phi|$. As soon as $|\nabla\Phi|$ exceeds K, we reinitialize the level set function.

The proof of the theorem above is split into three propositions. We first introduce the following lemmas, and then state the three propositions.

LEMMA 3.2. *Let ϕ be a smooth function and surface $\Gamma = \{(x,y) : \phi(x,y) = 0\}$ be the zero level set of ϕ. Then for any function $u, v \in L^2(\Omega)$, the modified surface gradient operator ∇_Γ satisfies*

$$|\nabla_\Gamma u|^2 \leqslant |\nabla u|^2 \leqslant |\nabla u|_\delta^2.$$

Proof. If we use the original $\nabla_\Gamma \phi$, then it is a projection of $\nabla \phi$ on the tangent plane, the inequality means the length of the projection is smaller than the original vector, which is true automatically. But the modified operator is no longer projection. However, the modified operator satisfies

$$\nabla_\Gamma u = \nabla u - (\nabla u \cdot \vec{n}_\delta)\vec{n}_\delta,$$

Therefore, we have

$$\begin{aligned} |\nabla_\Gamma u|^2 &= \nabla_\Gamma u \cdot \nabla_\Gamma u \\ &= (\nabla u - (\nabla u \cdot \vec{n}_\delta)\vec{n}_\delta) \cdot (\nabla u - (\nabla u \cdot \vec{n}_\delta)\vec{n}_\delta) \\ &= \nabla u \cdot \nabla u - 2(\nabla u \cdot \vec{n}_\delta)^2 + (\nabla u \cdot \vec{n}_\delta)^2(\vec{n}_\delta \cdot \vec{n}_\delta) \\ &= |\nabla u|^2 - (\nabla u \cdot \vec{n}_\delta)^2(2 - |\vec{n}_\delta|^2) \\ &\leqslant |\nabla u|^2 \leqslant |\nabla u|_\delta^2. \end{aligned}$$

by the fact $0 \leqslant |\vec{n}_\delta|^2 \leqslant |\vec{n}|^2 = 1$. In addition, we can verify this is also true for the discretized solution. ∎

LEMMA 3.3. *We have the following inequalities*

$$\begin{aligned} &\|D^2 u\|_2^2 \leqslant \|\Delta u\|_2^2 \leqslant 2\|D^2 u\|_2^2, \\ &\|D^3 u\|_2^2 \leqslant \|\nabla \Delta u\|_2^2 \leqslant 3\|D^3 u\|_2^2, \\ &\|D^2 u\|_2^2 \leqslant \|\nabla u\|_2 + \|D^3 u\|_2. \end{aligned}$$

Proof. Integrate by parts for $\|\Delta u\|_2^2$ and we obtain

$$\|\Delta u\|_2^2 = \int (u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2)dxdy = \int (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)dxdy.$$

The second part can be verified in a similar way. For the third part, we have

$$\int u_{xx}^2 dxdy = -\int u_x u_{xxx} dxdy \leqslant \frac{1}{2}\left(\int u_x^2 dxdy + \int u_{xxx}^2 dxdy\right).$$

Do the same to $\int u_{xy}^2 dxdy$ and $\int u_{yy}^2 dxdy$, we can come to the conclusion. ∎

LEMMA 3.4. *For any u, there exist some constant $C = C(\Omega)$ such that*

$$\|D^2 u\|_4^2 \leqslant C\|\nabla u\|_\infty \|D^3 u\|_2.$$

Proof. By Gagliardo-Nirenberg inequality as in [31], for any f we have

$$\|Df\|_4^2 \leqslant C\|f\|_\infty \|D^2 f\|_2.$$

By taking $f = \Phi_x$ and $f = \Phi_y$ we obtain the inequality. ∎

Proposition 3.5. *(Consistency) Under the same assumptions as in Theorem 3.1, the numerical scheme (3.7) is consistent with equation (2.4) with local truncation error* $\|\tau^k\|_{-1} = O(dt)$.

Proof. The local truncation error is defined as

$$\tau^k = \frac{\phi^{k+1}-\phi^k}{dt} + \beta\Delta^2(\phi^{k+1}-\phi^k) - |\nabla\phi^k|_\delta \nabla_\Gamma\big(g(h_\delta^k)\nabla_\Gamma h_\delta^k\big). \tag{3.10}$$

Taking the Taylor series of ϕ at kdt and assuming that $\|\phi_{tt}\|_{-1}$, $\|\nabla\Delta\phi_t\|_2$, $\|\nabla\phi\|_\infty$ and $\|\phi_t\|_{-1}$ are bounded, we obtain that

$$\|\tau^k\|_{-1} = O(dt).$$

thus the local truncation error is first order in time. ∎

Proposition 3.6. *(Stability) Under the same assumptions as Theorem 3.1 and assume* $\|\nabla\Phi^k\|_\infty \leqslant K$ *for all* $kdt \leqslant T$, *then the numerical solution* Φ^k *satisfies*

$$\|\nabla\Phi^k\|_2^2 + dtK_1\|\nabla\Delta\Phi^k\|_2^2 \leqslant e^{K_2T}\big(\|\nabla\Phi_0\|_2^2 + dtK_1\|\nabla\Delta\Phi_0\|_2^2\big),$$

for some constant K_1, K_2.

Proof. We multiply (3.7) with $\Delta\Phi^{k+1}$ and integrate over Ω, then we obtain

$$\frac{\langle\Phi^{k+1},\Delta\Phi^{k+1}\rangle - \langle\Phi^k,\Delta\Phi^{k+1}\rangle}{dt} + \beta\big(\langle\Delta^2\Phi^{k+1},\Delta\Phi^k\rangle - \langle\Delta^2\Phi^{k+1},\Delta\Phi^k\rangle\big) = -\left\langle \mathrm{div}_{\mathbf{\Gamma}}\big(g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k\big), |\nabla\Phi^k|_\delta\Delta\Phi^{k+1}\right\rangle. \tag{3.11}$$

Integrate by parts for both sides, then we obtain

$$\frac{\langle\nabla\Phi^{k+1},\nabla\Phi^{k+1}\rangle - \langle\nabla\Phi^k,\nabla\Phi^{k+1}\rangle}{dt} + \beta\big(\|\nabla\Delta\Phi^{k+1}\|_2^2 - \langle\nabla\Delta\Phi^{k+1},\nabla\Delta\Phi^k\rangle\big) = -\left\langle g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k, \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})\right\rangle.$$

Applying Cauchy's inequality we obtain

$$\langle\Phi^{k+1},\Phi^k\rangle \leqslant \frac{1}{2}\big(\|\Phi^{k+1}\|_2^2 + \|\Phi^k\|_2^2\big).$$

Consequently, we have

$$\|\Phi^{k+1}\|_2^2 - \langle\Phi^{k+1},\Phi^k\rangle \leqslant \frac{1}{2}\big(\|\Phi^{k+1}\|_2^2 - \|\Phi^k\|_2^2\big).$$

Similarly, we have

$$\|\Delta\Phi^{k+1}\|_2^2 - \langle\Delta\Phi^{k+1},\Delta\Phi^k\rangle \leqslant \frac{1}{2}\big(\|\Delta\Phi^{k+1}\|_2^2 - \|\Delta\Phi^k\|_2^2\big).$$

Therefore, we obtain the following inequality by lemma 3.2:

$$\begin{aligned}\frac{\|\nabla\Phi^{k+1}\|_2^2 - \|\nabla\Phi^k\|_2^2}{2dt} &+ \frac{\beta}{2}\big(\|\nabla\Delta\Phi^{k+1}\|_2^2 - \|\nabla\Delta\Phi^k\|_2^2\big)\\ &= -\langle g(\mathbf{h}_\delta^k)|\nabla\Phi^k|_\delta\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k, \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})/|\nabla\Phi^k|_\delta\rangle\\ &\leqslant \frac{1}{2\varepsilon}\big\|g(\mathbf{h}_\delta^k)|\nabla\Phi^k|_\delta\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k\big\|_2^2 + \frac{\varepsilon}{2}\big\|\nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})/|\nabla\Phi^k|_\delta\big\|_2^2\\ &\leqslant \frac{L}{2\varepsilon}\big\||\nabla\Phi^k|_\delta\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k\big\|_2^2 + \frac{\varepsilon}{2}\big\|\nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})/|\nabla\Phi^k|_\delta\big\|_2^2\\ &\leqslant \frac{L}{2\varepsilon}\big\||\nabla\Phi^k|_\delta\nabla\mathbf{h}_\delta^k\big\|_2^2 + \frac{\varepsilon}{2}\big\|\nabla(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})/|\nabla\Phi^k|_\delta\big\|_2^2.\end{aligned} \tag{3.12}$$

Then we estimate $\nabla \mathbf{h}_\delta^k$. Similar to the original level set representation of mean curvature h, the modified h_δ has the following representation.

$$\mathbf{h}_\delta^k \;=\; \nabla\cdot\big(\frac{\nabla\Phi^k}{|\nabla\Phi^k|_\delta}\big) = \frac{\Delta\Phi^k}{|\nabla\Phi^k|_\delta} - \frac{(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3},$$

and

$$\begin{aligned}|\nabla\Phi^k|_\delta\nabla\mathbf{h}_\delta^k \;&=\; |\nabla\Phi^k|_\delta\nabla\big(\frac{\Delta\Phi^k}{|\nabla\Phi^k|_\delta} - \frac{(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3}\big)\\ &=\; \nabla\Delta\Phi^k - \frac{\Delta\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^2} - \frac{2\nabla^2\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^2}\\ &-\; \frac{\nabla\Phi^k\nabla(\nabla^2\Phi^k)\nabla\Phi^k}{|\nabla\Phi^k|_\delta^2} + \frac{3(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^4}.\end{aligned}$$

In addition, we have

$$\nabla(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})/|\nabla\Phi^k|_\delta \;=\; \nabla\Delta\Phi^{k+1} + \Delta\Phi^{k+1}\frac{\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^2}.$$

By the fact that $|\nabla\Phi^k|_\delta \geqslant \delta$ for all k, we have

$$|\nabla\mathbf{h}_\delta^k||\nabla\Phi^k|_\delta \leqslant |\nabla\Delta\Phi^k| + \frac{6|\nabla^2\Phi^k|^2}{\delta} + |D^3\Phi^k|,$$

and

$$|\nabla(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})|/|\nabla\Phi^k|_\delta \leqslant |\nabla\Delta\Phi^{k+1}| + \frac{1}{\delta}|\Delta\Phi^{k+1}||D^2\Phi^k|.$$

Consequently,

$$L\big\||\nabla\Phi^k|_\delta\nabla\mathbf{h}_\delta^k\big\|_2^2 \leqslant C_1\|D^3\Phi^k\|_2^2 + C_2\|D^2\Phi^k\|_4^4,$$

and

$$\begin{aligned}\big\|\nabla(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})/|\nabla\Phi^k|_\delta\big\|_2^2 \;&\leqslant\; \|\nabla\Delta\Phi^{k+1}\|_2^2 + C_4\|\Delta\Phi^{k+1}\nabla^2\Phi^k\|_2^2\\ &\leqslant\; C_3\|D^3\Phi^{k+1}\|_2^2 + C_4\|D^2\Phi^{k+1}\|_4^4 + C_4\|D^2\Phi^k\|_4^4.\end{aligned}$$

Therefore, we have the following estimate with lemma 3.3 and 3.4:

$$L\big\||\nabla\Phi^k|_\delta\nabla\mathbf{h}_\delta^k\big\|_2^2 \leqslant C_5\|D^3\Phi^k\|_2^2 \leqslant C_6\|\nabla\Delta\Phi^k\|_2^2, \tag{3.13}$$

and

$$\big\|\nabla(|\nabla\Phi^k|_\delta\Delta\Phi^{k+1})/|\nabla\Phi^k|_\delta\big\|_2^2 \leqslant C_7\|\nabla\Delta\Phi^{k+1}\|_2^2 + C_8\|\nabla\Delta\Phi^k\|_2^2. \tag{3.14}$$

By plugging into (3.12) we obtain

$$\|\nabla\Phi^{k+1}\|_2^2 + (\beta - C_7\varepsilon)dt\|\nabla\Delta\Phi^{k+1}\|_2^2 \leqslant \|\nabla\Phi^k\|_2^2 + (\beta + C_6 + C_8\varepsilon)dt\|\nabla\Delta\Phi^k\|_2^2. \tag{3.15}$$

By choosing $\varepsilon = \frac{\beta}{2C_7}$ we obtain

$$\begin{aligned}\|\nabla\Phi^{k+1}\|_2^2 + \frac{\beta}{2}dt\|\nabla\Delta\Phi^{k+1}\|^2 \leqslant&\|\nabla\Phi^k\|_2^2 + \big(\beta + C_6 + \frac{\beta C_8}{2C_7}\big)dt\|\nabla\Delta\Phi^k\|^2\\ \leqslant&(1 + (1 + 2C_6/\beta + C_8/C_7)dt)(\|\nabla\Phi^k\|_2^2 + \frac{\beta}{2}dt\|\nabla\Delta\Phi^k\|^2).\end{aligned}$$

Taking $K_1 = \beta/2$ and $K_2 = (1 + 2C_6/\beta + C_8/C_7)dt$, then we have the following inequality by induction.

$$\begin{aligned}\|\nabla\Phi^k\|_2^2 + K_1 dt\|\nabla\Delta\Phi^k\|^2 \leqslant&(1+K_2dt)(\|\nabla\Phi_{k-1}\|_2^2 + K_1dt\|\nabla\Delta\Phi_{k-1}\|^2)\\ \leqslant&(1+K_2dt)^k(\|\nabla\Phi_0\|_2^2 + K_1dt\|\nabla\Delta\Phi_0\|^2)\\ \leqslant&e^{K_2T}(\|\nabla\Phi_0\|_2^2 + K_1dt\|\nabla\Delta\Phi_0\|^2).\end{aligned}$$

which gives the boundedness of the numerical solution. ◻

In this proposition we make the assumption that $\|\nabla\Phi^k\|_\infty \leqslant K$. This assumption is reasonable when we are using a level set method for curve evolution problems. We should keep the level function smooth to avoid any undesired singulary, thus we always reinitialize the level set function once $\|\nabla\Phi^k\|_\infty$ reaches the given upper bound K. The assumption $\|\nabla\Phi^k\|_\infty \leqslant K$ is used in the proof of convergence. The convergence of the discrete solution to the continuous solution as $dt \to 0$ is included in the following proposition.

Proposition 3.7. *(Convergence) Under the same assumptions as in Theorem 3.1, the discretization error $e^k = \phi^k - \Phi^k$ with $kdt \leqslant T$ for a fixed $T > 0$ satisfies*

$$\|\nabla e^k\|_2^2 + K_1dt\|\nabla\Delta e^k\|_2^2 \leqslant Te^{K_2T}\cdot Cdt^2$$

for some constants C, K_1, K_2.

Proof. Subtracting (21) from (22) we obtain

$$\frac{e^{k+1}-e^k}{dt}+\beta\Delta^2e^{k+1}-\beta\Delta^2e^k = -|\nabla\phi^k|_\delta\nabla_\Gamma\big(g(h_\delta^k)\nabla_\Gamma h_\delta^k\big)+|\nabla\Phi^k|_\delta\nabla_{\mathbf{\Gamma}}\big(g(\mathbf{h}_\delta^k)\nabla_\Gamma\mathbf{h}_\delta^k\big)+\tau^k.$$

We use the same technique as the proof of proposition 2. Multiplying both sides with Δe^{k+1} and integrating by parts, we obtain

$$\begin{aligned}&\frac{\|\nabla e^{k+1}\|_2^2 - \langle\nabla e^k, \nabla e^{k+1}\rangle}{dt} + \beta\big(\|\nabla\Delta e^{k+1}\|_2^2 - \langle\nabla\Delta e^{k+1}, \nabla\Delta e^k\rangle\big)\\ =&-\big\langle g(h_\delta^k)\nabla_\Gamma h_\delta^k, \nabla_\Gamma(|\nabla\phi^k|_\delta\Delta e^{k+1})\big\rangle + \big\langle g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k, \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1})\big\rangle\\ &+ \langle\nabla\Delta^{-1}\tau^k, \nabla\Delta e^{k+1}\rangle\\ =&\langle\nabla\Delta^{-1}\tau^k, \nabla\Delta e^{k+1}\rangle - \big\langle g(h_\delta^k)\nabla_\Gamma h_\delta^k - g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k, \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1})\big\rangle\\ &- \big\langle g(h_\delta^k)\nabla_\Gamma h_\delta^k, \nabla_\Gamma(|\nabla\phi^k|_\delta\Delta e^{k+1}) - \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1})\big\rangle.\end{aligned}$$

For the difference terms above, we split them into five terms:

$$\begin{aligned}&g(h_\delta^k)\nabla_\Gamma h_\delta^k - g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k\\ =&(g(h_\delta^k) - g(\mathbf{h}_\delta^k))\nabla_\Gamma h_\delta^k + g(\mathbf{h}_\delta^k)(\nabla_\Gamma - \nabla_{\mathbf{\Gamma}})h_\delta^k + g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}(h_\delta^k - \mathbf{h}_\delta^k)\\ =&(I)\ +\ (II)\ +\ (III),\\ &\nabla_\Gamma(|\nabla\phi^k|_\delta\Delta e^{k+1}) - \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1})\\ =&(\nabla_\Gamma - \nabla_{\mathbf{\Gamma}})(|\nabla\phi^k|_\delta\Delta e^{k+1}) + \nabla_{\mathbf{\Gamma}}(|\nabla\phi^k|_\delta\Delta e^{k+1} - |\nabla\Phi^k|_\delta\Delta e^{k+1})\\ =&(IV)\ +\ (V).\end{aligned}$$

Now we estimate $(I)-(V)$. First we estimate $h_\delta^k - \mathbf{h}_\delta^k$:

$$\begin{aligned}h_\delta^k - \mathbf{h}_\delta^k =&\Big(\frac{\Delta\phi^k}{|\nabla\phi^k|_\delta} - \frac{(\nabla\Phi^k)^T\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^3}\Big) - \Big(\frac{\Delta\Phi^k}{|\nabla\Phi^k|_\delta} - \frac{(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3}\Big)\\ =&\Big(\frac{\Delta\phi^k}{|\nabla\phi^k|_\delta} - \frac{\Delta\Phi^k}{|\nabla\Phi^k|_\delta}\Big) - \Big(\frac{(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^3} - \frac{(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3}\Big).\end{aligned}$$

With the fact $|\nabla\phi|_\delta > \delta$, $|\nabla\Phi|_\delta > \delta$ and the assumption $\|\partial^\alpha\phi\|_\infty \leqslant K$ for $|\alpha| \leqslant 3$, $\|\nabla\Phi\|_\infty \leqslant K$, we obtain

$$\begin{aligned}
\Big|\frac{\Delta\phi^k}{|\nabla\phi^k|_\delta} - \frac{\Delta\Phi^k}{|\nabla\Phi^k|_\delta}\Big| &= \Big|\frac{\Delta\phi^k - \Delta\Phi^k}{|\nabla\Phi^k|_\delta} + \frac{\Delta\phi^k(|\nabla\Phi^k|_\delta - |\nabla\phi^k|_\delta)}{|\nabla\phi^k|_\delta|\nabla\Phi^k|_\delta}\Big| \\
&\leqslant \frac{|\Delta\phi^k - \Delta\Phi^k|}{|\nabla\Phi^k|_\delta} + \frac{|\Delta\phi^k|\big||\nabla\Phi^k|_\delta - |\nabla\Phi^k|_\delta\big|}{|\nabla\phi^k|_\delta|\nabla\Phi^k|_\delta} \\
&\leqslant \frac{|\Delta e^k|}{|\nabla\Phi^k|_\delta} + \frac{|\Delta\phi^k||\nabla e^k|}{|\nabla\Phi^k|_\delta|\nabla\Phi^k|_\delta} \\
&\leqslant \frac{|\Delta e^k|}{\delta} + \frac{|\nabla e^k||D^2\phi^k|}{\delta^2},
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\frac{(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^3} - \frac{(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3}\Big| \\
&\leqslant \Big|\frac{(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3}\Big| + \Big|\frac{(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^3} - \frac{(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k}{|\nabla\Phi^k|_\delta^3}\Big| \\
&\leqslant \Big|\frac{(\nabla\phi^k - \nabla\Phi^k)^T\nabla^2\phi^k\nabla\phi^k}{|\nabla\Phi^k|_\delta^3}\Big| + \Big|\frac{(\nabla\Phi^k)^T\nabla^2\phi^k(\nabla\phi^k - \nabla\Phi^k)}{|\nabla\Phi^k|_\delta^3}\Big| + \Big|\frac{(\nabla\Phi^k)^T(\nabla^2\phi^k - \nabla^2\Phi^k)\nabla\phi^k}{|\nabla\Phi^k|_\delta^3}\Big| \\
&\qquad + |(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k|\Big|\frac{(|\nabla\phi^k|_\delta - |\nabla\Phi^k|_\delta)(|\nabla\phi^k|_\delta^2 + |\nabla\phi^k|_\delta|\nabla\Phi^k|_\delta + |\nabla\Phi^k|_\delta^2)}{|\nabla\phi^k|_\delta^3|\nabla\Phi^k|_\delta^3}\Big| \\
&\leqslant \Big|\frac{(\nabla e^k)^T\nabla^2\phi^k\nabla\phi^k}{|\nabla\Phi^k|_\delta^3}\Big| + \Big|\frac{(\nabla\Phi^k)^T\nabla^2\phi^k\nabla e^k}{|\nabla\Phi^k|_\delta^3}\Big| + \Big|\frac{(\nabla\Phi^k)^T\nabla^2 e^k\nabla\phi^k}{|\nabla\Phi^k|_\delta^3}\Big| \\
&\qquad + |(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k|\Big|\frac{(|\nabla e^k|_\delta)(|\nabla\phi^k|_\delta^2 + |\nabla\phi^k|_\delta|\nabla\Phi^k|_\delta + |\nabla\Phi^k|_\delta^2)}{|\nabla\phi^k|_\delta^3|\nabla\Phi^k|_\delta^3}\Big| \\
&\leqslant C(|\Delta e^k| + |\nabla e^k|). \qquad (3.16)
\end{aligned}$$

Therefore, we have the following inequalities:

$$|h_\delta^k - \mathbf{h}_\delta^k| \leqslant C_1(|\Delta e^k| + |\nabla e^k|),$$

and

$$\begin{aligned}
|g(h_\delta^k) - g(\mathbf{h}_\delta^k)| &= |g'(h^*)(h_\delta^k - \mathbf{h}_\delta^k)| \\
&\leqslant LC_1(|\Delta e^k| + |\nabla e^k|),
\end{aligned}$$

and

$$\begin{aligned}
|(\nabla_\Gamma - \nabla_{\mathbf{\Gamma}})h_\delta^k| &= \Big|\Big(\big(I - \frac{\nabla\phi^k\nabla(\phi^k)^T}{|\nabla\phi^k|_\delta^2}\big) - \big(I - \frac{\nabla\Phi^k\nabla(\Phi^k)^T}{|\nabla\Phi^k|_\delta^2}\big)\Big)\nabla h_\delta^k\Big| \\
&= \Big|\Big(\frac{\nabla\phi^k\nabla(\phi^k)^T}{|\nabla\phi^k|_\delta^2} - \frac{\nabla\Phi^k\nabla(\Phi^k)^T}{|\nabla\Phi^k|_\delta^2}\Big)\nabla h_\delta^k\Big| \\
&\leqslant C_2|\nabla e^k|.
\end{aligned}$$

To estimate $|\nabla(h_\delta^k - \mathbf{h}_\delta^k)|$, recall that we represent $\nabla\mathbf{h}_\delta$ by

$$\begin{aligned}
\nabla\mathbf{h}_\delta &= \frac{\nabla\Delta\Phi^k}{|\nabla\Phi^k|_\delta} - \frac{\Delta\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{2\nabla^2\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} \\
&- \frac{\nabla\Phi^k\nabla(\nabla^2\Phi^k)\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} + \frac{3(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^5},
\end{aligned}$$

and we have the corresponding one for ∇h_δ^k. Then the estimate goes as follows.

$$|\nabla(h_\delta^k - \mathbf{h}_\delta^k)| \leqslant \Big|\frac{\nabla\Delta\Phi^k}{|\nabla\Phi^k|_\delta} - \frac{\nabla\Delta\phi^k}{|\nabla\phi^k|_\delta}\Big| + \Big|\frac{\Delta\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{\Delta\phi^k\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^3}\Big|$$
$$+ 2\Big|\frac{\nabla^2\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{\nabla^2\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3}\Big| + \Big|\frac{\nabla\Phi^k\nabla(\nabla^2\Phi^k)\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{\nabla\phi^k\nabla(\nabla^2\phi^k)\nabla\phi^k}{|\nabla\phi^k|_\delta^3}\Big|$$
$$+ 3\Big|\frac{(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^5} - \frac{(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^5}\Big|.$$

With similar analysis as above, we have

$$\Big|\frac{\nabla\Delta\Phi^k}{|\nabla\Phi^k|_\delta} - \frac{\nabla\Delta\phi^k}{|\nabla\phi^k|_\delta}\Big| \leqslant C(|\nabla\Delta e^k| + |\nabla e^k|),$$

and

$$\Big|\frac{\nabla\Phi^k\nabla(\nabla^2\Phi^k)\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{\nabla\phi^k\nabla(\nabla^2\phi^k)\nabla\phi^k}{|\nabla\phi^k|_\delta^3}\Big| \leqslant C(|D^3e^k| + |\nabla e^k|).$$

In addition, we use the fact that $\Delta\Phi^k = \Delta\phi^k - \Delta e^k$ and $\nabla^2\Phi^k = \nabla^2\phi^k - \nabla^2 e^k$ to estimate the other three difference terms. Here we use again the assumption that the derivatives of ϕ is bounded. We have

$$\Delta\Phi^k\nabla^2\Phi^k = (\Delta\phi^k - \Delta e^k)(\nabla^2\phi^k - \nabla^2 e^k)$$
$$= \Delta\phi^k\nabla^2\phi^k - \Delta\phi^k\nabla^2 e^k - \Delta e^k\nabla^2\phi^k + \Delta e^k\nabla^2 e^k.$$

Therefore, we obtain the following estimate.

$$\Big|\frac{\Delta\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{\Delta\phi^k\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^3}\Big|$$
$$\leqslant \Big|\frac{(\Delta\Phi^k\nabla^2\Phi^k - \Delta\phi^k\nabla^2\phi^k)\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3}\Big| + \Big|\frac{\Delta\phi^k\nabla^2\phi^k(\nabla\Phi^k - \nabla\phi^k)}{|\nabla\Phi^k|_\delta^3}\Big|$$
$$+ \Big|\frac{\Delta\phi^k\nabla^2\phi^k\nabla\phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{\Delta\phi^k\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^3}\Big|$$
$$\leqslant C(|D^2e^k|^2 + |D^2e^k| + |\nabla e^k|).$$

Similarly, we have

$$\Big|\frac{\nabla^2\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3} - \frac{\nabla^2\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^3}\Big| \leqslant C(|D^2e^k|^2 + |D^2e^k| + |\nabla e^k|),$$

and

$$\Big|\frac{(\nabla\Phi^k)^T\nabla^2\Phi^k\nabla\Phi^k\nabla^2\Phi^k\nabla\Phi^k}{|\nabla\Phi^k|_\delta^5} - \frac{(\nabla\phi^k)^T\nabla^2\phi^k\nabla\phi^k\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|_\delta^5}\Big| \leqslant C(|D^2e^k|^2 + |D^2e^k| + |\nabla e^k|).$$

Thus we obtain the following estimate:

$$|\nabla(h_\delta^k - \mathbf{h}_\delta^k)| \leqslant C_3(|D^3e^k| + |D^2e^k| + |D^2e^k|^2 + |\nabla e^k|).$$

Consequently, the estimation for (I), (II), (III) are

$$|(I)| \leqslant |g(h_\delta^k) - g(\mathbf{h}_\delta^k)||\nabla h_\delta^k| \leqslant C_4(|\Delta e^k| + |\nabla e^k|),$$

$$|(II)| = |g(\mathbf{h}_\delta^k)(\nabla_\Gamma - \nabla_{\mathbf{\Gamma}})h_\delta^k| \leqslant LC_1|(\nabla_\Gamma - \nabla_{\mathbf{\Gamma}})h_\delta^k| \leqslant C_5|\nabla e^k|,$$

$$\begin{aligned}|(III)| &= |g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}(h_\delta^k - \mathbf{h}_\delta^k)| \leqslant L|\nabla(h_\delta^k - \mathbf{h}_\delta^k)| \\ &\leqslant C_6(|D^3e^k| + |D^2e^k| + |D^2e^k|^2 + |\nabla e^k|).\end{aligned}$$

Estimating (IV) and (V) is slightly different. We have

$$\nabla(|\nabla\phi^k|_\delta\Delta e^{k+1}) \ = \ \frac{\nabla\Delta e^{k+1}}{|\nabla\phi^k|_\delta} + \Delta e^{k+1}\frac{\nabla^2\phi^k\nabla\phi^k}{|\nabla\phi^k|^3}.$$

Therefore,

$$\begin{aligned}|(IV)| &= |(\nabla_\Gamma - \nabla_{\mathbf{\Gamma}})(|\nabla\phi^k|_\delta\Delta e^{k+1})| \\ &= \Big|\Big(\frac{\nabla\phi^k\nabla(\phi^k)^T}{|\nabla\phi^k|_\delta^2} - \frac{\nabla\Phi^k\nabla(\Phi^k)^T}{|\nabla\Phi^k|_\delta^2}\Big)\nabla(|\nabla\phi^k|_\delta\Delta e^{k+1})\Big| \\ &\leqslant C_7|\nabla e^k||\nabla(|\nabla\phi^k|_\delta\Delta e^{k+1})| \\ &\leqslant C_8|\nabla e^k|(|D^3e^{k+1}| + |D^2e^{k+1}|), \\ |(V)| &= |\nabla_{\mathbf{\Gamma}}(|\nabla\phi^k|_\delta\Delta e^{k+1} - |\nabla\Phi^k|_\delta\Delta e^{k+1})| \\ &\leqslant |\nabla(|\nabla\phi^k|_\delta\Delta e^{k+1} - |\nabla\Phi^k|_\delta\Delta e^{k+1})| \\ &\leqslant |\nabla\Delta e^{k+1}|\big||\nabla\phi^k|_\delta - |\nabla\Phi^k|_\delta\big| + |\Delta e^{k+1}||\nabla(|\nabla\phi^k|_\delta - |\nabla\Phi^k|_\delta)| \\ &\leqslant C_9(|D^3e^{k+1}||\nabla e^k| + |D^2e^{k+1}||D^2e^k| + |D^2e^{k+1}||\nabla e^k|).\end{aligned}$$

In addition, we use $\nabla^2\Phi^k = \nabla^2\phi^k - \nabla^2e^k$ again and obtain the following estimate

$$\begin{aligned}|\nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1})| &\leqslant |\nabla(|\nabla\Phi^k|_\delta\Delta e^{k+1})| \\ &\leqslant |\nabla\Phi^k|_\delta|\nabla\Delta e^k| + |\Delta e^k|\frac{|\nabla^2\Phi^k\nabla\Phi^k|}{|\nabla\Phi^k|_\delta^2} \\ &\leqslant C_{10}(|D^3e^{k+1}| + |D^2e^{k+1}| + |D^2e^{k+1}||D^2e^k|) \\ &\leqslant C_{10}(|D^3e^{k+1}| + |D^2e^{k+1}| + |D^2e^{k+1}|^2 + |D^2e^k|^2).\end{aligned}$$

Thus we obtain the following estimate:

$$\begin{aligned}|g(h_\delta^k)\nabla_\Gamma h_\delta^k - g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k| &\leqslant C_{11}(|D^3e^k| + |D^2e^k| + |D^2e^k|^2 + |\nabla e^k|) \\ |\nabla_\Gamma(|\nabla\phi^k|_\delta\Delta e^{k+1}) - \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1})| &\leqslant C_{12}(|D^3e^{k+1}||\nabla e^k| + |D^2e^{k+1}||D^2e^k| + |D^2e^{k+1}||\nabla e^k|) \\ &\leqslant C_{12}(|D^3e^{k+1}| + |D^2e^{k+1}|)(|D^2e^k| + |\nabla e^k|).\end{aligned}$$

Consequently applying lemma 3.3 and 3.4, and the fact $|\nabla e^k| \leqslant |\nabla\phi^k| + |\nabla\Phi^k| \leqslant 2K$, we obtain

$$\begin{aligned}&- \langle g(h_\delta^k)\nabla_\Gamma h_\delta^k - g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k, \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1})\rangle \\ \leqslant\ & \langle|g(h_\delta^k)\nabla_\Gamma h_\delta^k - g(\mathbf{h}_\delta^k)\nabla_{\mathbf{\Gamma}}\mathbf{h}_\delta^k|, |\nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1})|\rangle \\ \leqslant\ & C_{13}\langle|D^3e^{k+1}| + |D^2e^{k+1}| + |D^2e^{k+1}|^2 + |D^2e^k|^2, |D^3e^k| + |D^2e^k| + |D^2e^k| + |\nabla e^k|\rangle \\ \leqslant\ & C_{13}\varepsilon(\|D^3e^{k+1}\|_2^2 + |D^2e^{k+1}|_2^2 + |D^2e^{k+1}|_4^4 + |D^2e^k|_4^4) \\ & \qquad + C_{13}/\varepsilon(|D^3e^k|_2^2 + |D^2e^k|_2^2 + |D^3e^k|_4^4 + |\nabla e^k|_2^2) \\ \leqslant\ & C_{14}\varepsilon(\|D^3e^{k+1}\|_2^2 + |\nabla e^{k+1}|_2^2 + \|D^3e^k\|_2^2 + |\nabla e^k|_2^2) + C_{14}/\varepsilon(|D^3e^k|_2^2 + |\nabla e^k|_2^2) \\ =\ & C_{14}\varepsilon(\|D^3e^{k+1}\|_2^2 + |\nabla e^{k+1}|_2^2) + (C_{14}\varepsilon + C_{14}/\varepsilon)(|D^3e^k|_2^2 + |\nabla e^k|_2^2),\end{aligned}$$

and similarly

$$\begin{aligned}&|(g(h_\delta^k)\nabla_\Gamma h_\delta^k, \nabla_\Gamma(|\nabla\phi^k|_\delta\Delta e^{k+1}) - \nabla_{\mathbf{\Gamma}}(|\nabla\Phi^k|_\delta\Delta e^{k+1}))|\\ \leqslant & C_{15}\varepsilon(\|D^3e^{k+1}\|_2^2 + |\nabla e^{k+1}|_2^2) + C_{15}/\varepsilon(|D^3e^k|_2^2 + |\nabla e^k|_2^2).\end{aligned}$$

Now we come to the following estimate

$$\begin{aligned}&\frac{\|\nabla e^{k+1}\|_2^2 - \langle\nabla e^k, \nabla e^{k+1}\rangle}{dt} + \beta\big(\|\nabla\Delta e^{k+1}\|_2^2 - \langle\nabla\Delta e^{k+1}, \nabla\Delta e^k\rangle\big)\\ \leqslant & C_{16}\varepsilon(\|\nabla e^{k+1}\|_2^2 + \|\nabla\Delta e^{k+1}\|_2^2) + (C_{16}\varepsilon + C_{16}/\varepsilon)(\|\nabla e^k\|_2^2 + \|\nabla\Delta e^k\|_2^2) + \frac{1}{\varepsilon}\|\tau^k\|_{-1}^2,\end{aligned}$$

where ε is an arbitrary constant. Applying Cauchy's inequality to the left hand side we obtain

$$\begin{aligned}&(1 - C_{16}\varepsilon dt)\|\nabla e^{k+1}\|_2^2 + (\beta - C_{16}\varepsilon)dt\|\nabla\Delta e^{k+1}\|_2^2\\ \leqslant\ & (1 + C_{16}/\varepsilon dt)\|\nabla e^k\|_2^2 + (\beta + C_{16}\varepsilon + C_{16}/\varepsilon)dt\|\nabla\Delta e^k\|_2^2 + 1/\varepsilon\|\tau^k\|_{-1}^2.\end{aligned}$$

We take $\varepsilon = \frac{\beta}{2C_{16}}$. Then we can take $K_1 = 1 + \beta$ and $K_2 = \beta + C_{16}\varepsilon + C_{16}/\varepsilon$, as long as dt is small enough, we have

$$\|\nabla e^{k+1}\|_2^2 + K_1dt\|\nabla\Delta e^{k+1}\|_2^2 \leqslant (1 + K_2dt)(\|\nabla e^k\|_2^2 + K_1dt\|\nabla\Delta e^k\|_2^2) + \frac{\beta}{C_{10}}dt\|\tau^k\|_{-1}^2.$$

By induction, we obtain the following estimate:

$$\|\nabla e^k\|_2^2 + K_1dt\|\nabla\Delta e^k\|_2^2 \leqslant Te^{K_2T}\cdot Cdt^2.$$

We can see that $\|\nabla e^k\|_2$ converges with first order in time. □

When using a level set method for curve evolution problems, the existence of corners break the smoothness of the level set function. However, as in the previous discussion, this corner preserving model generalizes the LCIS equation, for which the corners are known to be in the infinitesimal sense. In Bertozzi and Greer [3] it has been proved in one dimension that the solutions of LCIS equations are smooth and never develop corners in finite time. We conjecture that it is also true for our model, as long as the curve has no self-intersections, although this has not been proven. In addition, our goal is to evolve the curve, which only involves a small neighborhood of the zero level set, so we only need to compute the corner preserving term of equation (2.5) in a narrow band around the zero level set of ϕ. This can save computational time. We may reinitialize ϕ to be the signed distance function, but not necessarily. According to our analysis, we impose an upper bound K for $|\nabla\Phi^k|$. If $|\nabla\Phi^k|$ exceeds K, reinitialization is required.

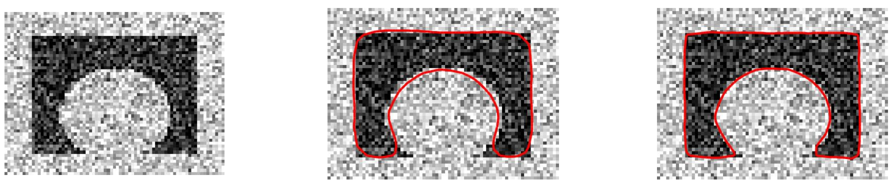
In two dimension images the operator $(1+dt\cdot\beta\Delta^2)^{-1}$ can be computed using the Fast Fourier Transform (FFT) very easily and efficiently. The evolution is still somewhat slow when comparing with Chan-Vese model due to the time step restriction. The Chan-Vese model only requires $dt \sim dx$ while our method requires $dt \sim dx^2$. In fact we can take advantage of fast methods for Chan-Vese by first solving Chan-Vese to steady state and using this as the initial guess for our method. Since we start with a initial guess that is close to the final result, the reinitialization process during the level set evolution is optional.

The full algorithm is:

Step 0.	Solve Chan-Vese model and obtain the steady state ϕ.
Step 1.	Initialize the level set function ϕ to the signed distance function.
Step 2.	Compute c_1 and c_2 for equation (2.5) and the Chan-Vese energy term.
Step 3.	Compute the corner preserving term in a narrow band around the zero level set of ϕ and set it to be 0 in other places. Usually we choose the narrow band as points within 3 or 4 grid size to the zero level set.
Step 4.	Update ϕ with equation (2.5).
Step 5.	Reinitialize the level set function ϕ to be the signed distance function if $\lvert\nabla\Phi^k\rvert$ exceeds K. Repeat step 2 until convergence.

4. Numerical Results. In this section we show some numerical results for image segmentation with the equation (2.5). Although we employ semi-implicit schemes, the spatial discretization of the nonlinear high order equation may impose additional time step restrictions. Usually we take $dt \sim dx^2$. For faster convergence, we do not directly solve equation (2.5) with a random initialization, but we start from the steady state of Chan-Vese method and then solve equation (2.5). The time step for equation (2.5) is chosen as $dt = .1dx^2$, while the time step for preprocessing with Chan-Vese method is $dt = .1dx$. As for the computational time, the regular Chan-Vese method takes 2 seconds and our methods takes 21 seconds for the building image in Figure 4.2 of size 128×110 using C++. For the hyperspectral image in Figure 4.3 of size 100×80 below, the Chan-Vese method takes 4 seconds and our methods takes 35 seconds.

Figure 4.1 shows the segmentation of a simple shape. (I) is the originally image. (II) shows the segmentation with equation (2.5) and (III) shows the segmentation without corner preserving term. Since the noise is strong in this image, the length regularization term has to be chosen large to avoid the local minima and small noisy pieces. We can see the corners are much better kept with the corner preserving term.



(I) Initial image (II) Segmentation without corner (III) Segmentation with corner

FIG. 4.1. *Comparison of Segmentation with and without corner term on a simple shape.*

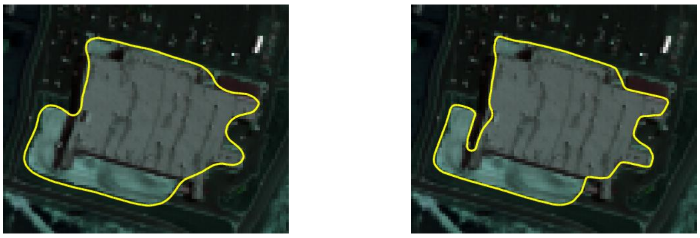
Figure 4.2 shows the segmentation of a building from Google maps. This is a 3-band color image. To avoid detecting the pieces on the roof, we have to use a strong regularization. The segmentation with equation (2.5) is better than the segmentation without the corner preserving term. And we also see that the two pieces enclosed by the building are also captured by the level set based segmentation method.

Figure 4.3 shows the segmentation of a Walmart building from a hyperspectral image with 163 bands. We can see that our approach also works for this high dimensional data. Here we can use the evolution as specified in equation (2.6). Note that

(I) Segmentation without corner (II) Segmentation with corner

FIG. 4.2. *Comparison of Segmentation with and without corner term on a building.*

we perform a simple binary segmentation which includes both the building and part of the ground. The corner preserving method more accurately segments these two features.



(I) Segmentation without corner (II) Segmentation with corner

FIG. 4.3. *Comparison of Segmentation with and without corner term on a building.*

5. Conclusion. In this paper we propose a modification of the Chan-Vese model. Motivated by the low curvature image simplifier, we add a corner preserving term to the Chan-Vese model following a method developed in Droske and Bertozzi [17] for image snakes. With the new model we can capture the sharp corners in the image while we can still manage the complex topology. To solve the high order nonlinear equation, we employ the numerical technique of adding a bilaplacian term and using semi-implicit schemes, which improves the time step restriction from $dt \sim dx^4$ to $dt \sim dx^2$. We also prove the stability and convergence of the semi-implicit time stepping scheme. We validate our model by numerical tests on color and hyperspectral images. The numerical results also show that this new model is robust to noise. One issue is that due to the nonlinearity and high order, we have to use smaller time steps when comparing with the original Chan-Vese model. Future work could involve faster numerical schemes to speed up this method, or the application of this model to surface representation and reconstruction as in [17].

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REFERENCES

[1] G. Bellettini and M. Paolini. Anisotropic motion by mean curvature in the context of finsler geomety. *Hokkaido Math. J.*, 25:537–566, 1996.

[2] M. Bertalmio, L.T. Cheng, G. Sapiro, and S. Osher. Variational problems and partial differential equations on implicit surfaces. *J. Comput. Phys.*, 174:759–780, 2001.

[3] A. Bertozzi and J.B. Greer. Low curvature image simplifiers: global regularity of smooth solutions and laplacian limiting schemes. *Comm. Pure Appl. Math.*, 57:764–790, 2004.

[4] A. Bertozzi, N. Ju, and H. Lu. A biharmonic modified forward time stepping method for fourth order nonlinear diffusion equations. *DCDS*, 29(4):1367–1391, 2001.

[5] M. Burger. Numerical simulation of anisotropic surface diffusion with curvature-dependent energy. *J. Comput. Phys.*, pages 602–625.

[6] V. Caselles, F. Catte, T. Coll, and F. Dibos. A geometric model for active contours in image processing. *Numer. Math.*, 66:1–31, 1993.

[7] V. Caselles and B. Coll. Snakes in movement. *SIAM J. Numer. Anal.*, 33:2445–2456, 1996.

[8] V. Caselles, R. Kimmel, and G. Sapiro. Geodesic active contours. *International Journal of Computer Vision*, 22:61–79, 1997.

[9] A. Chambolle. An algorithm for total variation minimization and applications. *Journal of Mathematical Imaging and Vision*, 19:89–97, 2004.

[10] T. Chan, S. Esedoglu, and M. Nikolova. Algorithms for finding global minimizers of image segmentation and denoising models. *SIAM J. Appl. Math.*, 66:1632–1648, 2006.

[11] T. Chan, B. Sandberg, and L. Vese. Active contours without edges for vector-valued images. *Journal of Visual Communication and Image Representation*, 11:130–141, 2000.

[12] T. Chan and L. Vese. An active contour model without edges. *Scale-Space Theories in Computer Vision. Second International Conference*, 1999.

[13] D.L. Chopp and J.A. Sethian. Motion by intrinsic laplacian of curvature. *Interfaces and Free Boundaries*, 1:1–18, 1999.

[14] U. Clarenz. The wulff-shape minimizes an anisotropic willmore functional. *Interfaces and Free Boundaries*, 6:351–359, 2004.

[15] K. Deckelnick and G. Dziuk. Discrete anisotropic curvature flow of graphs. *Mathematical Modelling and Numerical Analysis*, 33:1203–1222, 1999.

[16] K. Deckelnick, G. Dziuk, and C.M. Elliott. Fully discrete simi-implicit second order splitting for anisotropic surface diffusion of graphs. *SIAM J. Numer. Anal.*, 43:1112–1138, 2005.

[17] M. Droske and A. Bertozzi. Higher-order feature-preserving geometric regularization. *SIAM J. Image Sci.*, 3:21–51, 2010.

[18] Elsey and S. Esedoglu. Analogue of the total variation denoising model in the context of geometry processing. *UCLA CAM report 07-31*, 2007.

[19] S. Esedoglu and S. Osher. Decomposition of images by the anisotropic Rudin-Osher-Fatemi model. *Comm. Pure Appl. Math.*, 57:1609–1626, 2004.

[20] M. Kass, A. Witkin, and D. Terzopoulos. Snakes: Active contour models. *International Journal of Computer Vision*, 1:321–331, 1998.

[21] S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum, and A. Yezzi. Gradient flows and geometric active contour models. *Proceedings of the Fifth International Conference on Computer Vision, ICCV*, pages 810–815, 1995.

[22] J. Morel and S. Solimini. Variational methods in image segmentation. *Progress in Nonlinear Differential Equations and Their Applications*, 14, 1995.

[23] D. Mumford and J. Shah. Optimal approximation by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.*, 42:577–685, 1989.

[24] S. Osher and J. A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulation. *J. Comput. Phys.*, 79:12–49, 1988.

[25] Y. Pan, J.D. Birdwell, and S. Djouadi. Efficient implementation of the Chan-Vese models without solving PDEs. *IEEE 8th Workshop on Multimedia Signal Processing*, pages 3350–354, 2006.

[26] P. Perona and J. Malik. Scale space and edge detection using anisotropic diffusion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12:629–639, 1990.

[27] D. Salac and W. Lu. A local semi-implicit level-set method for interface motion. *J. Sci. Comput.*, 35:330–349, 2008.

[28] G. Sapiro. Vector snakes: a geometric framework for color, texture and multi-scale image segmentation. *Proc. IEEE Int. Conf. Image Proc*, 1996.

[29] C. Schoenlieb and A. Bertozzi. Unconditionally stable schemes for higher order inpainting. *to appear in Comm. Math. Sci.*

[30] P. Smereka. Semi-implicit level set methods for curvature and surface diffusion motion. *J. Sci.*

Comput., 19:439–456, 2003.

[31] T.Tao. Nonlinear dispersive equations: local and global analysis. *CBMS regional series in mathematics*, 2006.

[32] J. Tumblin and G. Turk. A boundary hierarchy for detail-preserving contrast reduction. *Proceedings of the SIGGRAPH annual conference on Computer Graphics*, pages 83–90, 1999.

[33] J. Ye. Applications of variational models in geometric problems. *Ph.D. Thesis, Department of Mathematics, UCLA*, 2009.