Mathematical Models and Methods in Applied Sciences

LOCAL EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A PDE MODEL FOR CRIMINAL BEHAVIOR

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The analysis of criminal behavior with mathematical tools is a fairly new idea, but one which can be used to obtain insight on the dynamics of crime. In a recent work [34] Short *et al.* developed an agent-based stochastic model for the dynamics of residential burglaries. This model produces the right qualitative behavior, that is, the existence of spatio-temporal collections of criminal activities or "hotspots," which have been observed in residential burglary data. In this paper we prove local existence and uniqueness of solutions to the continuum version of this model, a coupled system of partial differential equations, as well a continuation argument. Furthermore, we compare this PDE model with a generalized version of the Keller-Segel model for chemotaxis as a first step to understanding possible conditions for global existence vs. blow-up of the solutions in finite time.

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1. Introduction

The study of crime hardly needs motivation, it is a phenomena that affects all individuals. The city of Los Angeles, nicknamed the "Gang Capital of the Nation," is of particular interest. Violent and non-violent crimes from burglaries to drive-by-shootings have affected the citizens of this city since the beginning of the 20th century. One of the most frequently occurring crimes is residential burglaries, a crime which will affect most people at some point. The observation that residential burglaries are not spatially homogeneously distributed and that certain neighborhoods have more propensity to crime than others led Short *et al.* to study the dynamics of residential burglary *hotspots*[34]. A *hotspot* is a spatio-temporal aggregation of criminal occurrences and the understanding how they evolve can be extremely useful. For example, it can help the police force mobilize their resources optimally. This would ideally lead to the reduction or even eliminations of these crime hotspots. A theoretical understanding of dynamics of hotspots would help predict how these hotpots will change and thus aid law enforcement agencies fight crime.

Short *et al.* modeled the dynamics of hotspots using an agent-based statistical model based on the 'broken window' sociological effect. [38] The idea of the 'broken window' effect is that crime in an area leads to more crime. It has been observed in the residential burglary data that houses which are burglarized have an increased probability of being burglarized again for some period of time after the initial burglary. This increased probability of burglary also affects neighboring houses and is referred to as the 'repeat near-repeat effect' [2, 25, 26, 27]. The model is based on the assumption that criminal agents are walking randomly on a two-dimensional lattice and committing burglaries when encountering an opportunity. Furthermore, there is an attractiveness value assigned to every house, which refers to how easily the house can be burgled without negative consequences for the criminal agent. The criminal agents, in addition to walking randomly, have a biased movement toward areas of high attractiveness values and move with a speed inversely proportional to the value in their current position. Let $A(x,t)$ and $\rho(x,t)$ be the attractiveness value and the criminal density at position x and time t respectively, then the continuum limit of the agent-based model gives the following PDE model:

$$\frac{\partial A}{\partial t} = \eta\Delta A - A + A\rho + A^o, \tag{1.1a}$$

$$\frac{\partial \rho}{\partial t} = \Delta\rho - 2\nabla\cdot[\rho\nabla\chi(A)] + \overline{B} - A\rho; \tag{1.1b}$$

where $\chi(A) = \log(A)$. A formal derivation of this model can be found in [34]. From (1.1) we observe that criminal agents are being created at a constant rate $\overline{B}$ and are removed from the model when a burglary is committed. In essence, the number of burglaries being committed at time t and location x is given by the $A(x,t)\rho(x,t)$. Furthermore, the attractiveness value increases with each burglary. As we will discuss later the system (1.1) can be seen as a nonlinear version of the Keller-Segel model for chemotaxis with growth and decay. The Keller-Segel model is a reaction-diffusion system that models the movement of some mobile species which is being influenced by an external chemo-attractant[5, 9, 12, 22, 37, 35]. In the Keller-Segel model literature the function $\chi(A)$ is referred to as the sensitivity function. Various forms of the sensitivity function have been analyzed including $\log(A)$ and A [30,33]. For these cases global existence has been proved in one-dimension [8,32]. Furthermore, in two-dimension global existence has been proved for small enough initial mass of the cell density [4,6]. It is important to note that these models do not include growth or decay. Although the logarithmic sensitivity function has been analyzed most of the research done on the Keller-Segel model has been for $\chi(A) = A$. Recall that the model (1.1) is the continuum limit of an discrete agent-based model. In the discrete model the probability of an agent moving from node s to node n is given by the ratio of the attractiveness value at node n over the sum of attractiveness values of the neighboring nodes of node s. This gives the logarithmic sensitivity function we see in (1.1). Therefore, it makes sense for us to analyze the more complicated sensitivity function. In fact, we will see later that the term $1/A$ in the advective term helps prevent blow-up.

From the numerical analysis performed in [34] this model seems to have appropriate qualitative properties, *i.e.* existence of hotspots. However, to show that this model is truly robust the unique existence of a solution, which does not blow up in finite time, is essential. The main result of this paper is the local existence of a solution in addition to a continuation argument, which gives a necessary and sufficient conditions for global existence. Assuming that the criminals entering the city and the criminals leaving are approximately the same we consider no-flux boundary condition in a bounded domain $\Omega \subset \mathbb{R}^2$:

$$\frac{\partial A}{\partial \nu}|_{\partial\Omega} = 0 \quad and \quad \frac{\rho}{A}\nabla A \cdot \vec{\nu}|_{\partial\Omega} = 0; \tag{1.2}$$

where ν is the outer normal vector. The initial conditions are given by:

$$\begin{aligned} A(0,x) &= A_0(x), \\ \rho(0,x) &= \rho_0(x). \end{aligned} \tag{1.3}$$

Outline. In Section 2 we state notation and existing theory to be used for the proof of the main result. Simultaneously, we give an outline of the proof. In Section 3 we prove the existence and uniqueness of a family of solutions to a regularized version of the original model. Following, in Section 4 we look at *a-priori* higher-order energy estimates that enable us to pass to the limit and prove the final result. Section 5 is devoted to proving a continuation argument. In Section 6 we look at a generalized version of the Keller-Segel model for chemotaxis and compare it with the PDE model for residential burglaries. In this section we also prove a blow-up argument for a modified residential burglary model. We conclude this paper in Section 7 with a final discussion.

2. Notation and Proof Outline of Main Result

We begin this section by establishing the notation that will be used throughout the paper. The proof of the main result follows the techniques used in [31] for the Navier-Stokes Equation in 3-D (see also [36] Taylor for symmetric hyperbolic systems). In the Keller-Segel literature there are two principal methods used to prove global existence of solutions to various versions of the model[23]. The first one involves finding L^∞ estimates for the advection term. The second method involves finding a Lyapunov function. Both of these methods use fixed point theory to obtain local solutions. Since we do not know of the existence of a Lyapunov function for (1.1) our method is more closely related to the first method mentioned. We use an abstract version of Picard's Theorem for ODEs to obtain a local solution to (1.1). We will see that global existence will end up being dependent on some L^∞ estimates.

2.1. *Notation*

We have a initial-boundary value problem with no-flux boundary conditions. For simplicity assume that our domain is a square. This problem can be mapped into the periodic problem with symmetry on a domain four times the size of the original domain. This is true provided A^o and the initial data satisfy reflection symmetry, in which case the model preserves symmetry. Hence, from now on we work with periodic boundary conditions and $\Omega = \mathbb{T}^2$ unless otherwise specified. It is useful to define the following notation:

$$\int v dx = \int_\Omega v dx,$$
$$\|v\|_0^2 = \int_\Omega v^2 dx.$$

Furthermore, for a multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$, we define the $H^m(\Omega)$-norm as follows:

$$\|v\|_m = \left(\sum_{|\alpha| \le m} \|D^\alpha v\|_0^2 \right)^{\frac{1}{2}}.$$

Finally, we define the spaces with their corresponding norms to be used:

- For X a Banach Space with norm $\|\cdot\|_X$, $C([0,T];X)$ is the space of continuous functions mapping $[0,T]$ into X. This space has the following norm:
$$\|v\|_{C([0,T];X)} := \sup_{0\le t\le T} \|v\|_X .$$
- $L_\infty(0,T;X)$ is the space of functions such that $v(t) \in X$ for $a.e.\ t \in (0,T)$ has finite norm:
$$\|v\|_{L_\infty(0,T;X)} := ess \sup_{t\in(0,T)} \|v(t)\|_X .$$
- $L^2(0,T;X)$ is the space of functions such that $v(t) \in X$ for $a.e.\ t \in (0,T)$ with finite norm:
$$\|v\|_{L^2(0,T;X)} := \left(\int_0^T \|v(t)\|_X^2 \, dt \right)^{\frac{1}{2}}.$$

Definition 2.1. The space $C^{weak}([0,T]; H^s(\Omega))$ denotes continuity on the interval $[0,T]$ with values in the weak topology of H^s. In other words, for any fixed $\Phi \in H^s$,

$(\Phi, u(t))_s$ is a continuous scalar function on $[0,T]$. The inner-product of H^s is given by:

$$(u,v)_s = \sum_{\alpha \leq s} \int D^\alpha u \cdot D^\alpha v dx. \tag{2.1}$$

The Hilbert Spaces we will be working on for most of the time is:

$$V^m = \{(u,v) \in H^m(\Omega) \times H^m(\Omega)\}. \tag{2.2}$$

Since we are working extensively with different bounds and the constants are not always important, we introduce the notation $A \lesssim B$ to mean that there exists a positive constant c such that $A \leq cB$. This notation will be used when the constants are irrelevant and become tedious.

2.2. *Main Result and Outline of its Proof*

Our main contribution is to prove local existence and uniqueness of solutions to the system (1.1). More precisely, we prove the following theorem.

Theorem 2.1 (Local Existence of Solutions to the PDE Residential Burglaries Model). *Given initial conditions* $(A_0(x), \rho_0(x)) \in V^m$ *for* $m > 3$ *such that* $A_0(x) > A^o$ *there exists a positive time,* $T > 0$*, such that* $A, \rho \in C([0,T]; C^2(\Omega)) \cap C^1([0,T]; C(\Omega))$ *form a unique solution to* (1.1) *on the time interval* $[0,T]$.

We first modify the system (1.1) by regularizing it, for the purpose of bounding differential operators in Sobolev Spaces. This is useful because finding a family of solutions to the regularized system is straightforward. Given $v \in L^p(\mathbb{T}^2)$ for $1 \leq p \leq \infty$ we define the mollification of v by

$$J_\epsilon v(x) = \sum_{k \in \mathbb{Z}^2} \hat{v}(k) e^{-\epsilon^2 |k|^2 + 2\pi i k \cdot x}, \tag{2.3}$$

where $\hat{v}(k) = \int_\Omega v(x) e^{-2\pi i k \cdot x} dx$. The mollified function, $J_\epsilon v_\epsilon$, has many useful properties, some of which are summarized in the following lemma. For more details we refer the reader to [3]. Furthermore, a proof can be found in [18]. We note that this is analogous to mollification by convolution with smooth functions in $\mathbb{R}^2$. The interested reader is referred to [16].

Lemma 2.1 (Properties of Mollifiers). *Let* J_ϵ *be a mollifier defined in* (2.3). *Then* $J_\epsilon v \in C^\infty$ *and has the following properties:*

(1) $\forall\ v\ \in C^1(\Omega)$ $J_\epsilon v \to v$ *uniformly and*

$$|J_\epsilon v|_\infty \leq |v|_\infty.$$

(2) Mollifiers commute with distribution derivatives,

$$D^\alpha J_\epsilon v = J_\epsilon D^\alpha v \quad \forall\ |\alpha| \leq m,\ v \in H^m.$$

(3) $\forall\, u, v \in L^2(\Omega)$,

$$\int_\Omega (J_\epsilon u) v dx = \int_\Omega (J_\epsilon v) u dx.$$

(4) $\forall\, v \in H^s(\Omega)$, *$J_\epsilon v$ converges to v in H^s and the rate of convergence in the H^{s-1} norm is linear in ϵ:*

$$\lim_{\epsilon \searrow 0} \|J_\epsilon v - v\|_s = 0,$$

$$\|J_\epsilon v - v\|_{s-1} \le C\epsilon \|v\|_s .$$

(5) $\forall\, v \in H^m(\Omega),\ \gamma,\ k \in Z^+ \cup 0,$ *and* $0 \le \epsilon \le 1$*:*

$$\|J_\epsilon v\|_{m+\gamma} \le \frac{c_{m\gamma}}{\epsilon^\gamma} \|v\|_m ,$$

$$\left|J_\epsilon D^k v\right|_\infty \le \frac{c_k}{\epsilon^{N/2+\gamma-k}} \|v\|_k .$$

Once the original system has been regularized it is easy to show that the assumptions of the Picard Theorem on a Banach Space are satisfied by the regularized model for any fixed $\epsilon > 0$. We now state this theorem along with a natural continuation theorem, since autonomous ODEs on a Banach Space have a natural continuation theorem. A proof of the following two theorems can be found in [19].

Theorem 2.2 (Picard Theorem on a Banach Space). *Let $O \subseteq \boldsymbol{B}$ be an open subset of a Banach Space $\boldsymbol{B}$, and let $F : O \to \boldsymbol{B}$ be a mapping satisfying:*

(1) $F(x)$ maps O to $\boldsymbol{B}$
(2) F is locally Lipschitz continuous i.e. for any $x \in O$ there exists $L > 0$ and an open neighborhood $U_x \subset O$ of x such that for all $x, \hat{x} \in U_x$ we have

$$\|F(x) - F(\hat{x})\|_{\boldsymbol{B}} \le L \|x - \hat{x}\|_{\boldsymbol{B}}$$

Then for any $x_o \in O$, there exist a time T such that the ODE

$$\frac{dx}{dt} = F(x), \quad x|_0 = x(0) \in O$$

has a unique local solution $x \in C^1((-T, T); O)$.

Theorem 2.3 (Continuation on a Banach Space). *Let $O \subseteq B$ be an open subset of a Banach Space B, and let $F : O \to B$ be a locally Lipschitz-continuous map. Then the unique solution $X \in C^1([0, T); O)$ to the autonomous ODE*

$$\frac{dx}{dt} = F(x), \quad x|_0 = x(0) \in O,$$

either exists globally in time, or $T < \infty$ and $X(t)$ leaves the open set O as $t \to T$.

We will see that the above theorem can be applied provided an appropriate functional framework is chosen. We use some calculus inequalities in the Sobolev Spaces to show that this theorem can be used to obtain a family of solutions which depend on the regularizing parameter ϵ. Refer to [31] for a proof of the following lemma in the case when $\Omega = \mathbb{R}^N$. The proof for the case when Ω is the torus follows exactly.

Lemma 2.2 (Calculus Inequalities in the Sobolev Spaces).

(1) $\forall\, m \in \mathbb{Z}^+ \cup\, 0$, *there exists* $c \geq 0$ *such that for all* $u, v \in L^\infty(\Omega) \cap H^m(\Omega)$*:*

$$\|uv\|_m \leq c\left\{|u|_\infty \|D^m v\|_0 + \|D^m u\|_0 |v|_\infty\right\},$$

$$\sum_{0\leq|\alpha|\leq m} \|D^\alpha(uv) - uD^\alpha v\|_0 \leq c\left\{|\nabla u|_\infty \left\|D^{m-1}v\right\|_0 + \|D^m u\|_0 |v|_\infty\right\}.$$

(2) $\forall\, s > \frac{N}{2}, H^s(\Omega)$ *is a Banach algebra. That is, there exists* $c > 0$ *such that for all* $u, v \in H^s(\Omega)$*:*

$$\|uv\|_s \leq c\, \|u\|_s \|v\|_s\,.$$

The next step is to pass to the limit as $\epsilon \to 0$. Energy estimates, which are independent of the regularizing parameter, are essential for this purpose.

3. Local Existence and Uniqueness of Solution to a Regularized Version of the Crime Model

We consider the following regularization of (1.1):

$$\frac{\partial A^\epsilon}{\partial t} = \eta J_\epsilon^2 \Delta A^\epsilon - A^\epsilon + \rho^\epsilon A^\epsilon + A^o, \tag{3.1a}$$

$$\frac{\partial \rho^\epsilon}{\partial t} = J_\epsilon(J_\epsilon \Delta\rho^\epsilon) - 2J_\epsilon[\nabla\cdot(\frac{\rho^\epsilon}{A^\epsilon}J_\epsilon\nabla A^\epsilon)] - \rho^\epsilon A^\epsilon + \overline{B}. \tag{3.1b}$$

This choice of regularization will become clear when we perform the energy estimate calculations. The goal of this section is to prove the local existence and uniqueness of solutions to the system (3.1) for fixed ϵ. Consider the function space for the solution to (3.1) to be the Banach Space V^2, $m = 2$ in (2.2), with norm $\|(A,\rho)\|_{V^2} := \|A\|_2 + \|\rho\|_2$.

Theorem 3.1 (Local Existence of solutions to the Regularized Residential Burglary Model). *For any* $\epsilon > 0$ *and initial conditions* $(A_0(x), \rho_0(x)) \in V^2$ *such that* $A_0(x) > A^o$ *there exists a solution,* $(A^\epsilon, \rho^\epsilon) \in C^1([0, T_\epsilon); V^2)$*, for some* $T_\epsilon > 0$*, to the regularized system* (3.1)*. Furthermore, the following energy estimate is satisfied,*

$$\frac{d}{dt}\|(A^\epsilon,\rho^\epsilon)\|_{V^2} \leq c_3 \|(A^\epsilon,\rho^\epsilon)\|_{V^2}^3 + c_2 \|(A^\epsilon,\rho^\epsilon)\|_{V^2}^2 + c_1 \|(A^\epsilon,\rho^\epsilon)\|_{V^2}; \tag{3.2}$$

where $c_1,\ c_2,$ *and* c_3 *are constants that depend only on* $\frac{1}{A^o}$, ϵ *and* η.

Proof. Define the map $F^\epsilon = [F_1^\epsilon, F_2^\epsilon] : O \subseteq V^2 \to X$. To use *Theorem 2.2* we need a suitable set O such that F^ϵ maps O to V^2, (*i.e.* $X = V^2$). Defined the function by:

$$F_1^\epsilon(A^\epsilon,\rho^\epsilon) = \eta J_\epsilon^2 \Delta A^\epsilon - A^\epsilon + \rho^\epsilon A^\epsilon + A^o, \tag{3.3a}$$

$$F_2^\epsilon(A^\epsilon,\rho^\epsilon) = J_\epsilon^2 \Delta \rho^\epsilon - 2J_\epsilon[\nabla\cdot(\frac{\rho^\epsilon}{A^\epsilon}J_\epsilon \nabla A^\epsilon)] - \rho^\epsilon A^\epsilon + \overline{B}. \tag{3.3b}$$

Hence, if $v^\epsilon = (A^\epsilon,\rho^\epsilon) \in V^2$, the original model reduces to an ODE in V^2.

$$\frac{dv^\epsilon}{dt} = F^\epsilon(v), \tag{3.4a}$$

$$v^\epsilon(0) = (A_0(x), \rho_0(x)). \tag{3.4b}$$

With this framework we can prove that the conditions of *Theorem 2.2* are satisfied. Let $v_i^\epsilon = (A_i^\epsilon, \rho_i^\epsilon) \in V^2$ $(i = 1, 2)$, we drop ϵ for notational convenience. By definition of the V^2-norm and F we have:

$$\|F(v_1) - F(v_2)\|_{V^2} = \|F_1(v_1) - F_1(v_2)\|_2 + \|F_2(v_1) - F_2(v_2)\|_2 .$$

After substituting (3.3) above and using *(5)* of *Lemma 2.1* and *(1)* of *Lemma 2.2* we obtain a suitable bound for F_1. Initially we have:

$$\|F_1(v_1) - F_1(v_2)\|_2 \leq \eta \left\|J_\epsilon^2 \Delta(A_1 - A_2)\right\|_2 + \|A_1 - A_2\|_2 + \|\rho_1 A_1 - A_2\rho_2\|_2 .$$

The last term in the above inequality will appear repeatedly and can be bounded using (2) of *Lemma 2.2* by:

$$\|\rho_1 A_1 - A_2 \rho_2\|_2 \lesssim \|\rho_2\|_2 \|A_1 - A_2\|_2 + \|A_1\|_2 \|\rho_1 - \rho_2\|_2 . \tag{3.5}$$

Using (3.5) we easily obtain the final estimate for F_1:

$$\|F_1(v_1) - F_1(v_2)\|_2 \lesssim (\frac{\eta}{\epsilon^2} + 1 + \|\rho_2\|_2)\|A_1 - A_2\|_2 + \|A_1\|_2 \|\rho_1 - \rho_2\|_2 . \tag{3.6}$$

For F_2 we only state the final bound, refer to Appendix A.1 for more detailed computations. If we define the open set

$$O = \left\{(u, v) \in V^2 : \left|\frac{1}{u}\right|_\infty < K_1, \|u\|_2 < L_1, \|v\|_2 < L_2\right\},$$

we obtain similar estimates for F_2. In particular, if $v_1,\ v_2 \in O$ then

$$\|F_2(v_1) - F_2(v_2)\|_2 \lesssim \widetilde{C_1}\|A_1 - A_2\|_2 + \widetilde{C_2}\|\rho_1 - \rho_2\|_2 ; \tag{3.7}$$

where,

$$\begin{aligned}
\widetilde{C_1} = &\frac{K_1}{\epsilon^3}\Big(\|\rho_1\|_2 + K_1\|A_1\|_1 |\rho_1|_\infty + K_1\|A_2\|_2\|\rho_2\|_2 + K_1^2\|A_2\|_2^2\|\rho_2\|_2\Big) \\
&+ \frac{K_1}{\epsilon}^3 \|A_1\|_1\|A_2\|_2\|\rho_2\|_2 + \frac{K_1}{\epsilon^2}|\rho_1|_\infty + \|\rho_2\|_2 , \\
\widetilde{C_2} = &\frac{1}{\epsilon^2} + \|A_1\|_2 + \frac{C_1^2}{\epsilon^3}\|A_2\|_2\|A_1\|_2(1 + K_1\|A_2\|_1 + K_1\|A_1\|_1) .
\end{aligned}$$

The important thing to note is that $\widetilde{C_1}$ and $\widetilde{C_2}$ depend only on $\|A_i\|_2$, $\|\rho_i\|_2$, ϵ, and K_1 for $i = 1, 2$. Combining (3.6) and (3.7) gives:

$$\|F(v_1)-F(v_2)\|_{V^2} \leq C(\eta, L_1, L_2, K_1, \epsilon) \|A_1-A_2\|_2 + C(L_1, L_2, K_1, \epsilon) \|\rho_1-\rho_2\|_2 . \tag{3.8}$$

Setting $A_2 = 0$ and $\rho_2 = 0$ we see that F does map O to V^2. Furthermore, $F : O \to V^2$ is locally Lipschitz therefore the conditions of *Theorem 2.2* are satisfied for fixed ϵ. Consequently, we obtain a family of unique local solutions to (3.1), $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$, such that $(A^\epsilon, \rho^\epsilon) \in C^1([0, T_\epsilon); V^2 \cap O)$. A careful look at the computations performed (see A.1) enables us to see that the constants in the above inequality are at most cubic in $\|(A, \rho)\|_{V^2}$. Once again, setting $A_2 = 0$ and $\rho_2 = 0$ in (3.8) from (3.4) we obtain the desired inequality (3.2). Note that the constants c_1, c_2 and c_3 depend solely on C_1, ϵ, and η. We by taking $K_1 = \frac{1}{A^o}$ we obtain the dependence on $1A^o$. $\square$

4. Local Existence and Uniqueness of Solution to Original Residential Burglary Model

In the previous section we successfully showed the unique existence of a solution to (3.1) on $[0, T_\epsilon)$ for fixed ϵ. The next step is to show that a subsequence of these solutions converge to a solution of the original system (1.1). To do this we need estimates that are independent of ϵ. The following section is devoted for this purpose.

4.1. *Energy Estimates*

From *Theorem* (3.1) we see that the time interval on which the solutions to (3.1) exist depend on ϵ. To be able to pass to the limit it is essential that we find a uniform time interval of existence. To obtain such an interval we look at energy estimates which are essential to show that the solution to (3.1) is in $C([0, T); V^m)$. We will see that provided m is chosen large enough then we obtain that the solution is classical. For simplicity from now on we denote $C_1 = \frac{1}{A^o}$.

Proposition 4.1 (Higher-Order Energy Estimates). *Let* $(A^\epsilon, \rho^\epsilon)$ *be a solution to the regularized system* (3.1) *with initial conditions* $(A^\epsilon(0), \rho^\epsilon(0)) \in V^m$*, where* V^m *is defined by* (2.2) *for* $m \geq 3$*, such that* $A_0(x) > A^o$*. If M is chosen large enough then* $E^\epsilon_m(t) = \frac{M}{2} \|A^\epsilon\|^2_m + \|\rho^\epsilon\|^2_m$ *satisfies the following differential inequalities:*

- *For* $m = 3$*:* $\frac{d}{dt} E^\epsilon_3(t) \lesssim C(M, C_1) (E^\epsilon_3)^{10} + C(A^o, \overline{B}, M).$
- *For* $m > 3$*:*

$$\frac{d}{dt} E^\epsilon_m(t) \lesssim C(M, C_1, |A|_\infty, |\rho|_\infty, |\nabla \rho^\epsilon|_\infty, |\nabla A^\epsilon|_\infty) E^\epsilon_m(t) + C(A^o, \overline{B}, M).$$

The proof of this proposition requires a sequence of lemmas. For these lemmas we let A^ϵ and ρ^ϵ be as in *Proposition 4.1.*

Lemma 4.1. *If M is an arbitrary constant then the following holds:*

$$\frac{M}{2}\frac{d}{dt}\left\|A^\epsilon\right\|_m^2 \lesssim -M\eta\left\|J^\epsilon\nabla A^\epsilon\right\|_m^2 + \frac{M}{2}\left\|A^o\right\|_0^2 + M\left(\left|\nabla A^\epsilon\right|_\infty + \left|\rho^\epsilon\right|_\infty + \left|A\right|_\infty\right)\left\|A^\epsilon\right\|_m^2 \\ + M\left(\left|\nabla A^\epsilon\right|_\infty + \left|A\right|_\infty\right)\left\|\rho^\epsilon\right\|_m^2. \tag{4.1}$$

Proof. Following standard procedure we first look at the time evolution equation of $\|A\|_m^2$. We drop ϵ for notational simplicity. Recalling the multi-index notation from Section 2.1 and using the chain rule we obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|A\right\|_m^2 = \sum_{|\alpha|\le m}\int (D^\alpha A)(D^\alpha A_t)dx.$$

For fixed α substitute in (3.1a) and obtain:

$$\begin{aligned}\int (D^\alpha A)(D^\alpha A_t)dx &= \int (D^\alpha A)D^\alpha(\eta J_\epsilon^2\Delta A - A + A\rho + A^o)dx \\ &= -\eta\left\|J^\epsilon D^\alpha\nabla A\right\|_0^2 - \left\|D^\alpha A\right\|_0^2 + \int (D^\alpha A)(D^\alpha A^o)dx \\ &\quad + \int (D^\alpha A)(D^\alpha(A\rho))dx.\end{aligned}$$

Note that the third term of the last equality will only contribute when $\alpha = \vec{0}$. For now consider the case $\alpha \neq \vec{0}$. The Cauchy-Schwarz inequality gives:

$$\int (D^\alpha A)(D^\alpha A_t)dx \le -\eta\left\|J^\epsilon D^\alpha\nabla A\right\|_0^2 - \left\|D^\alpha A\right\|_0^2 + \left\|D^\alpha A\right\|_0\left\|D^\alpha(A\rho)\right\|_0. \tag{4.2}$$

To simplify the computations we first look at the following claim. The derivation can be found in Appendix A.2 and uses part *(1)* of *Lemma 2.2.*
Claim 1:

$$\sum_{|\alpha|\le m}\left\|D^\alpha u\right\|_0\left\|D^\alpha(uv)\right\|_0 \lesssim \left(\left|\nabla u\right|_\infty + \left|u\right|_\infty + \left|v\right|_\infty\right)\left\|u\right\|_m^2 + \left(\left|\nabla u\right|_\infty + \left|u\right|_\infty\right)\left\|v\right\|_m^2.$$

Adding (4.2) over $|\alpha| \le m$:

$$\frac{M}{2}\frac{d}{dt}\left\|A\right\|_m^2 \le -M\eta\left\|J^\epsilon\nabla A\right\|_m^2 - M\left\|A\right\|_m^2 + M\|A^o\|_0\|A\|_0 + M\sum_{|\alpha|\le m}\left\|D^\alpha A\right\|_0\left\|D^\alpha(A\rho)\right\|_0.$$

Applying Cauchy-Schwarz Inequality to $M\|A^o\|_0\left\|A\right\|_0$ and *Claim 1* to the summation term gives the final result. □

Since the computations for ρ are more complicated we first look at the advection term.

Lemma 4.2. *For* $I_\alpha = \int \left\{ D^\alpha(J_\epsilon \nabla \rho^\epsilon) \cdot D^\alpha \left(\frac{\rho^\epsilon}{A^\epsilon} J^\epsilon \nabla A^\epsilon \right) \right\} dx$ *the following estimate holds for any* $0 < \delta < 1$*:*

$$2 \sum_{|\alpha| \le m} I_\alpha \lesssim \delta \left\| J_\epsilon \nabla \rho^\epsilon \right\|_m^2 + \frac{(C_1 C_2)^2}{\delta} \left\| J_\epsilon \nabla A^\epsilon \right\|_m^2 + \frac{1}{\delta} \left(C_1 \left| \nabla A^\epsilon \right|_\infty \right)^2 \left\| \rho^\epsilon \right\|_m^2$$
$$+ \frac{1}{\delta} \left(C_1 \left| \nabla \rho^\epsilon \right|_\infty + C_1^2 \left| \rho^\epsilon \right|_\infty \left| \nabla A^\epsilon \right|_\infty + \left| \rho^\epsilon \right|_\infty \sum_{k=0}^{m-1} \overline{C}_k C_1^{k+2} \left| \nabla A^\epsilon \right|_\infty^{k+1} \right)^2 \left\| A^\epsilon \right\|_m^2 .$$

The proof can be found in the Appendix A.2. We note that the power 10 in the energy inequality for the case when $m = 3$ in *Proposition 4.1* is comes from that fact that we are taking multiple derivatives of $1/A$.

Lemma 4.3.

$$\frac{1}{2} \frac{d}{dt} \left\| \rho^\epsilon \right\|_m^2 \lesssim (1 - \delta) \left\| J^\epsilon \nabla \rho^\epsilon \right\|_m^2 + \frac{1}{2} \left\| \overline{B} \right\|_0^2 + \frac{1}{2} \left\| \rho^\epsilon \right\|_0^2 + \beta_1 \left\| A^\epsilon \right\|_m^2 + \beta_2 \left\| \rho^\epsilon \right\|_m^2 \tag{4.3}$$
$$+ \frac{(C_1 C_2)^2}{\delta} \left\| J_\epsilon \nabla A^\epsilon \right\|_m^2 .$$

where,

- $\beta_1 = \left| \nabla \rho \right|_\infty + \left| \rho \right|_\infty + \frac{C_1}{\delta} \left(\left| \nabla \rho^\epsilon \right|_\infty + C_1 \left| \rho^\epsilon \right|_\infty \left| \nabla A^\epsilon \right|_\infty + \left| \rho^\epsilon \right|_\infty \sum_{k=0}^{m-1} \overline{C}_k C_1^{k+1} \left| \nabla A^\epsilon \right|_\infty^{k+1} \right)^2,$
- $\beta_2 = \left| \nabla \rho^\epsilon \right|_\infty + \left| A^\epsilon \right|_\infty + \left| \rho \right|_\infty + \frac{1}{\delta} C_1^2 \left| \nabla A \right|_\infty^2 .$

Proof. For fixed α substitute in (3.1b):

$$\begin{aligned} \int (D^\alpha \rho)(D^\alpha \rho_t)\, dx &= \int (D^\alpha \rho) D^\alpha \left(J_\epsilon^2 \Delta \rho - 2 J_\epsilon \nabla \cdot \left(\frac{\rho}{A} J^\epsilon \nabla A \right) - A\rho + \overline{B} \right) dx \\ &\le - \left\| J^\epsilon D^\alpha \nabla \rho \right\|_0^2 + \left\| D^\alpha \rho \right\|_0 \left\| D^\alpha \overline{B} \right\|_0 + \left\| D^\alpha \rho \right\|_0 \left\| D^\alpha (A\rho) \right\|_0 \\ &\quad + 2 \underbrace{\int D^\alpha (J_\epsilon \nabla \rho) \cdot D^\alpha \left(\frac{\rho}{A} J^\epsilon \nabla A \right) dx.}_{I_\alpha} \end{aligned}$$

Simply using *Lemma 4.2* and *Claim 1* we obtain the final estimate for ρ given by (4.3). □

Combining *Lemma 4.1* and *Lemma 4.3* gives the proof of *Proposition 4.1*.

Proof. (Proposition 4.1) Recalling that we have the estimate $\left| \rho \right|_\infty \le c \left\| \rho \right\|_2$ then $\left| \rho_0(x) \right|_\infty \le c L_2 =: C_2$. Combine (4.1) and (4.3) by first fixing $\delta < 1$ and then choosing $M > \frac{1}{\eta \delta} (C_1 C_2)^2$. In fact, if $\delta_1 = (1 - \delta) > 0$ and $\delta_2 = M \eta \delta - (C_1 C_2)^2 > 0$ then:

$$\frac{d}{dt} E_m(t) + \delta_1 \left\| J_\epsilon \nabla \rho^\epsilon \right\|_m^2 + \delta_2 \left\| J_\epsilon \nabla A^\epsilon \right\|_m^2 \le D_1 \left\| A^\epsilon \right\|_m^2 + D_2 \left\| \rho^\epsilon \right\|_m^2 + C(A^o, \overline{B}, M). \tag{4.4}$$

where,

- $C(A^o, \overline{B}, M) = \frac{M}{2} \|A^o\|_0^2 + \frac{1}{2} \|\overline{B}\|_0^2$,
- $D_1 = \beta_1 + M\left(|\nabla A^\epsilon|_\infty + |\rho^\epsilon|_\infty + |A|_\infty\right)$,
- $D_2 = \beta_2 + M\left(|\nabla A^\epsilon|_\infty + |A|_\infty\right)$.

Observe that the coefficients of $\|\rho^\epsilon\|_m^2$ and $\|A^\epsilon\|_m^2$ depend only on $|\nabla A^\epsilon|_\infty$, $|\nabla \rho^\epsilon|_\infty$, $|A^\epsilon|_\infty$, $|\rho^\epsilon|_\infty$, and C_1. From Sobolev embedding estimates we have $|\nabla u|_\infty \leq c\,\|u\|_3$; hence, it is natural to first consider the case $m = 3$. This case is useful to get an initial estimate of T from (4.4). Indeed, we obtain the desired result for this case:

$$\frac{d}{dt} E_3(t) \lesssim C\left(M, C_1\right)\left(E_3\right)^{10} + C(A^o, \overline{B}, M). \tag{4.5}$$

The power ten on E_3 in (4.5) comes from *Lemma 4.2*. Fortunately, this estimate is independent of the regularizing parameter ϵ. Hence, there exists a positive time, T, such that the H^3-norms of A and ρ are bounded on $[0, T]$. Considering the case where $m > 3$ gives the second desired inequality:

$$\frac{d}{dt} E_m(t) + \delta_1 \|J_\epsilon \nabla \rho^\epsilon\|_m^2 + \delta_2 \|J_\epsilon \nabla A^\epsilon\|_m^2 \lesssim C E_m(t) + C(A^o, \overline{B}, M), \tag{4.6}$$

with $C = C\left(M, C_1, |A|_\infty, |\rho|_\infty, |\nabla \rho^\epsilon|_\infty, |\nabla A^\epsilon|_\infty\right)$. $\square$

Remark 4.1. Note that for the above argument we needed $|\rho|_\infty < C_2$. Due to the Sobolev Embedding Theorem the L_∞-norm is controlled by the H^2-norm. From *Theorem 3.1* each $\epsilon > 0$ we know that $\|\rho^\epsilon\|_2 < L_2$ for $t \in [0, T_\epsilon)$. However, we know that $[0, T] \subset [0, T_\epsilon)$.

The bound on the higher-norms of the regularized solutions prove to be extremely useful in multiple ways. To begin with, all higher-order norms are bounded on $[0, T]$. Moreover, we know that there exists some $\tau > 0$ such that $A^\epsilon(x, t) \geq A^o$ for all $(x, t) \in \Omega \times [0, \tau]$ if $A^\epsilon(x, 0) > A^o$. Indeed, if we define $A_*^\epsilon = \min_{x \in \Omega} A^\epsilon(x, t)$ then we have a point-wise bound on its time derivative thanks *Proposition 4.1*. In fact, we know that:

$$\begin{aligned} \left|\frac{dA_*^\epsilon}{dt}\right| &\leq \eta\,|\Delta A^\epsilon|_\infty + |A^\epsilon + A^\epsilon \rho^\epsilon + A^o|_\infty \\ &\leq \eta\,\|\Delta A^\epsilon\|_2 + \|A^\epsilon + A^\epsilon \rho^\epsilon + A^o\|_2\,, \end{aligned}$$

where we need $A^\epsilon \in H^m$ for $m > 4$ to use *1* of *Lemma 2.1* and then Sobolev embedding estimates. Since $\|A^\epsilon\|_4$ is bounded independent of ϵ then $A_*^\epsilon > A^o$ on $[0, \tau]$ for some $\tau \in [0, T]$. For simplicity let $T = \min\{T, \tau\}$, from now on we interval $[0, T]$ to be the interval on which the higher-order norms are bounded and $A^\epsilon \geq A^o$. Now that we have a non-trivial interval on which all the higher-order norms are bounded we show that the family of solutions to the regularized system (3.1), $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon > 0}$, form a Cauchy sequence in the L^2-norm. This enables us to obtain the necessary limiting functions A, ρ, which are a solutions to (1.1).

Lemma 4.4. *The family of solutions* $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon > 0}$ *to* (3.1) *form a Cauchy sequence in* $C([0, T]; L^2(\Omega) \times L^2(\Omega))$. *In particular, there exists a constant* C *and a time* $T > 0$

such that for all ϵ and ϵ'

$$\sup_{0\le t\le T}\left\{\left\|A^\epsilon - A^{\epsilon'}\right\|_0 + \left\|\rho^\epsilon - \rho^{\epsilon'}\right\|_0\right\} \le C\max(\epsilon,\epsilon').$$

Proof. Let $(A^\epsilon,\rho^\epsilon)$ and $(A^{\epsilon'},\rho^{\epsilon'})$ solve their respective regularized systems (3.1) and satisfy the conditions of the lemma. Take the inner-product of $A^\epsilon - A^{\epsilon'}$ and $A^\epsilon_t - A^{\epsilon'}_t$.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|A^\epsilon{-}A^{\epsilon'}\right\|_0^2 &= \int\left(A^\epsilon{-}A^{\epsilon'}\right)\left(A^\epsilon_t{-}A^{\epsilon'}_t\right)dx\\ &= \int\left(A^\epsilon{-}A^{\epsilon'}\right)\left(\eta J_\epsilon^2\Delta A^\epsilon{-}\eta J_\epsilon^2\Delta A^{\epsilon'}\right)dx - \left\|A^\epsilon{-}A^{\epsilon'}\right\|_0^2\\ &\quad+\int\left(A^\epsilon{-}A^{\epsilon'}\right)\left(A^\epsilon\rho^\epsilon{-}A^{\epsilon'}\rho^{\epsilon'}\right)dx = I_1+I_2+I_3.\end{aligned}$$

Since I_2 has a negative sign it is not problematic. The other two terms can be easily dealt with using *(4)* of *Lemma 2.1*.

$$\begin{aligned}I_1 &= -\eta\left\|J_{\epsilon'}\nabla(A^\epsilon - A^{\epsilon'})\right\|_0^2 + \eta\int(J_\epsilon^2 - J_{\epsilon'}^2)\Delta A^\epsilon(A^\epsilon - A^{\epsilon'})dx\\ &\le -\eta\left\|J_{\epsilon'}\nabla(A^\epsilon - A^{\epsilon'})\right\|_0^2 + \eta\max(\epsilon,\epsilon')\left\|A\right\|_3\left\|A^\epsilon - A^{\epsilon'}\right\|_0.\end{aligned}$$

For the last term,

$$\begin{aligned}I_3 &= \int\rho^\epsilon(A^\epsilon - A^{\epsilon'})^2dx + \int A^{\epsilon'}\left(A^\epsilon - A^{\epsilon'}\right)\left(\rho^\epsilon - \rho^{\epsilon'}\right)dx\\ &\le \left(|\rho^\epsilon|_\infty + \frac{1}{2}|A^\epsilon|_\infty\right)\left\|A^\epsilon - A^{\epsilon'}\right\|_0^2 + \frac{1}{2}|A^\epsilon|_\infty\left\|\rho^\epsilon - \rho^{\epsilon'}\right\|_0^2.\end{aligned}$$

Combine these inequalities and return to the initial estimate to obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|A^\epsilon{-}A^{\epsilon'}\right\|_0^2 &\le \left(|\rho^\epsilon|_\infty{+}\frac{1}{2}|A^\epsilon|_\infty{-}1\right)\left\|A^\epsilon{-}A^{\epsilon'}\right\|_0^2 + \eta\max(\epsilon,\epsilon')\left\|A\right\|_3\left\|A^\epsilon{-}A^{\epsilon'}\right\|_0\\ &\quad+\frac{1}{2}|A^\epsilon|_\infty\left\|\rho^\epsilon{-}\rho^{\epsilon'}\right\|_0^2. \qquad (4.7)\end{aligned}$$

Perform a similar computation for ρ:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\rho^\epsilon - \rho^{\epsilon'}\right\|_0^2 &= \int\left(\rho^\epsilon - \rho^{\epsilon'}\right)\left(\rho^\epsilon_t - \rho^{\epsilon'}_t\right)dx\\ &= \int\left(\rho^\epsilon{-}\rho^{\epsilon'}\right)\left(J_\epsilon^2\Delta\rho^\epsilon{-}J_\epsilon^2\Delta\rho^{\epsilon'}\right)dx + \int\left(\rho^\epsilon{-}\rho^{\epsilon'}\right)\left(A^\epsilon\rho^\epsilon{-}A^{\epsilon'}\rho^{\epsilon'}\right)dx\\ &\quad+\int\left(\rho^\epsilon - \rho^{\epsilon'}\right)\left(J_\epsilon\left(\nabla\cdot\frac{\rho^\epsilon}{A^\epsilon}J_\epsilon\nabla A^\epsilon\right) - J_{\epsilon'}\left(\nabla\cdot\frac{\rho^{\epsilon'}}{A^{\epsilon'}}J_{\epsilon'}\nabla A^{\epsilon'}\right)\right)dx\\ &= F_1+F_2+F_3.\end{aligned}$$

The terms F_1 and F_2 are dealt with exactly as was done for the attractiveness value. F_3 is not as straight forward but it can be simplified using Cauchy-Schwarz inequality:

$$F_3 \leq \left(\left\| J_\epsilon \left(\nabla \cdot \frac{\rho^\epsilon}{A^\epsilon} J_\epsilon A^\epsilon \right) \right\|_0 + \left\| J_{\epsilon'} \left(\nabla \cdot \frac{\rho^{\epsilon'}}{A^{\epsilon'}} J_{\epsilon'} \nabla A^{\epsilon'} \right) \right\|_0 \right) \left\| \rho^\epsilon - \rho^{\epsilon'} \right\|_0 .$$

We can extract an ϵ at the expense of a higher-order norm and the loss of a mollifier. For example we have:

$$\begin{aligned} \left\| J_\epsilon \left(\nabla \cdot \frac{\rho^\epsilon}{A^\epsilon} J_\epsilon \nabla A^\epsilon \right) \right\|_0 &\leq \epsilon \left\| \frac{\rho^\epsilon}{A^\epsilon} J_\epsilon \nabla A^\epsilon \right\|_2 \\ &\lesssim \epsilon \left\{ \left| \frac{\rho^\epsilon}{A^\epsilon} \right|_\infty \left\| D^2 \nabla A^\epsilon \right\|_0 + \left| \nabla A^\epsilon \right|_\infty \left\| D^2 \left(\frac{\rho^\epsilon}{A^\epsilon} \right) \right\|_0 \right\} . \end{aligned}$$

From the proof of *Lemma 4.2*, refer to the inequality (A.3), the above inequality has a bound that depends only on $\|\rho^\epsilon\|_2$, $\|A^\epsilon\|_3$, and C_1. Define $v^2 = \left\| A^\epsilon - A^{\epsilon'} \right\|_0^2 + \left\| \rho^\epsilon - \rho^{\epsilon'} \right\|_0^2$. Since $\|A^\epsilon\|_3$ and $\|\rho^\epsilon\|_2$ are bounded on $[0, T]$ then we have the following differential inequality:

$$\frac{d}{dt} v \lesssim C(\max(\epsilon, \epsilon') + v).$$

Notice that the constant depends on C_1, $\|\rho^\epsilon\|_2$ and $\|A^\epsilon\|_3$. The above differential inequality gives $v(t) \leq e^{Ct} \left(v(0) + \max(\epsilon, \epsilon') \right) - \max(\epsilon, \epsilon')$. Since $(A^\epsilon, \rho^\epsilon)$ and $(A^{\epsilon'}, \rho^{\epsilon'})$ satisfy the same initial conditions we have that $v(0) = 0$, which implies:

$$\sup_{0 \leq t < T} v(t) \leq C \max(\epsilon, \epsilon'). \qquad \square$$

4.2. *Existence and Uniqueness of Solutions to the Original Residential Burglary Model*

We have all the tools to prove *Theorem* (2.1); however, we first state and prove the result for uniqueness of solutions. More precisely, if we assume that we have existence of a smooth enough solution to (1.1) then this solution must be unique.

Lemma 4.5 (Uniqueness of Smooth Solutions). *Let $(A_1, \rho_1), (A_2, \rho_2)$ be local-in-time solutions, with a common interval of existence $[0, T]$, to the system (1.1). Furthermore, suppose these solutions are smooth enough and with the same initial data in V^m, for $m \geq 3$, which satisfy the conditions stated in Theorem 2.1 then $A_1 = A_2$ and $\rho_1 = \rho_2$ on $[0, T]$.*

Proof. We consider the difference of both variables $u = A_1 - A_2$ and $v = \rho_1 - \rho_2$.

From (1.1) we can see that u and v satisfy the following system:

$$u_t = \eta\Delta u - u + \rho_1 u + A_2 v, \tag{4.8a}$$

$$v_t = \Delta v - 2\nabla\cdot\left(\frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2\right) - \rho_1 u - A_2 v. \tag{4.8b}$$

The time evolution of the L^2-norm of u multiplied by a constant M (the same M used in *Lemma 4.1*) satisfies the following inequality:

$$\frac{d}{dt}\frac{M}{2}\|u\|_0^2 \le -M\eta\|\nabla u\|_0^2 + M\left(|\rho_1|_\infty + \frac{1}{2}|A_2|_\infty - 1\right)\|u\|_0^2 + \frac{M}{2}|A_2|_\infty\|v\|_0^2. \tag{4.9}$$

The above inequality can be seen simply by taking the L^2-inner product of u_t and u. Substituting (4.8a) for u_t into this inner product and integrating by parts gives (4.9). The same is done for v. The following inequality holds:

$$\frac{d}{dt}\frac{1}{2}\|v\|_0^2 \lesssim C_1^2C_2^2\|\nabla u\|_0^2 + C\left(|\rho_1|_\infty, |\nabla A_2|_\infty\right)\|u\|_0^2 + C(|\rho_1|_\infty, |\nabla A_2|_\infty, |A_1|_\infty)\|v\|_0^2. \tag{4.10}$$

For detailed computations of the upper bound given by (4.10) refer to Appendix A.3. Define $F(t) = \frac{M}{2}\|u(t)\|_0^2 + \frac{1}{2}\|v(t)\|_0^2$, again choosing $M > \frac{1}{\eta}(C_1C_2)^2$ then from (4.9) and (4.10) we see that $F(t)$ satisfies the following ode:

$$\frac{dF(t)}{dt} \le C_M F(t). \tag{4.11}$$

In (4.11) the constant $C_M = C_M(M, |\rho_1|_\infty, |A_1|_\infty, |A_2|_\infty, |\nabla A_1|_\infty, C_1)$. We are set to apply a Grönwall's lemma [31]. Applying this lemma to (4.11) gives that $\sup_{0\le t\le T}\{F(t)\} \le F(0)e^{C_M T}$. All terms that compose C_M are bounded on the interval $[0,T]$. Since the two solutions satisfy the same initial conditions then $F(0) = 0$, which implies uniqueness of the solution. □

We now progress to the proof of the main result: *Theorem 2.1.*

Proof. From *Theorem 3.1* we have that given the initial conditions in the hypothesis of *Theorem 2.1*, there exists a family of solutions $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$ to the regularized problem (3.1). These solutions exist on the time interval $[0, T_\epsilon)$. The interval of existence depends on the regularizing parameter; however, from *Lemma 4.1* we know that the V^2-norm of the solutions are bounded independent of ϵ. This gives a uniform interval of existence $[0,T]$. Furthermore, from *Lemma 4.4* we conclude that there exist $A,\ \rho \in C([0,T]; L^2(\Omega))$ such that:

$$\sup_{0\le t\le T}\{\|A^\epsilon - A\|_0 + \|\rho^\epsilon - \rho\|_0\} \le C\epsilon.$$

Therefore, the solutions converge strongly in the low-norm. We state an interpolation lemma needed to show strong convergence in intermediate norms. This lemma offers a connection between *Lemma 4.1* and *Lemma 4.4* which leads to the desired result.

Lemma 4.6 (Interpolation in Sobolev Spaces). *Given $s \geq 0$, there exists a constant C_s so that for all $v \in H^s(\Omega)$, and $0 < s' < s$ the following inequality holds:*

$$\|v\|_{s'} \leq \|v\|_0^{1-s'/s} \|v\|_s^{s'/s}.$$

To use *Lemma 4.6* having strong convergence in the L^2-norm and some bounds on the higher norms is essential. For $m > 3$ we apply the above lemma to $\overline{A} = A^\epsilon - A$ and $\overline{\rho} = \rho^\epsilon - \rho$.

$$\begin{aligned} \sup_{0 \leq t \leq T} \left\{ \left\|\overline{A}\right\|_{m'} + \|\overline{\rho}\|_{m'} \right\} &\lesssim \left(\left\|\overline{A}\right\|_0^{1-m'/m} \left\|\overline{A}\right\|_m^{m'/m} + \|\overline{\rho}\|_0^{1-m'/m} \|\overline{\rho}\|_m^{m'/m} \right) \\ &\lesssim \left(\left\|\overline{A}\right\|_m^{m'/m} \epsilon^{1-m'/m} + \|\overline{\rho}\|_m^{m'/m} \epsilon^{1-m'/m} \right). \end{aligned}$$

The estimate (4.6) implies that A^ϵ, ρ^ϵ are uniformly bounded in H^m, for $m \geq 2$. Therefore, the above inequality implies strong convergence in $C([0,T], V^{m'})$. Taking m' to be larger than three implies strong convergence in $C([0,T], C^2(\Omega))$ due to the Sobolev Embedding Theorem [14]. Now, we simply need to verify that the limits A and ρ actually satisfy (1.1). Since $(A^\epsilon, \rho^\epsilon) \to (A, \rho)$ from (3.1) we see that A_t^ϵ converges to $\eta\Delta A - A + A\rho + A^o$ in $C([0,T], C(\Omega))$. Correspondingly, ρ_t^ϵ converge to $\nabla \cdot \left[\nabla\rho - 2\frac{\rho}{A}\nabla A\right] + \overline{B} - A\rho$. Finally, since $A_t^\epsilon \to A_t$ and $\rho_t^\epsilon \to \rho_t$ then A and ρ are classical solutions of (1.1). Since the solutions satisfy the smoothness requirements of *Lemma 4.5* they are unique. □

5. Continuation of the Solutions to the Residential Burglary Model

In the previous section we proved that if the initial data $(A(0,x), \rho(0,x)) \in V^m$ then there exists some positive time T, such that there exists a classical solution $(A(x,t), \rho(x,t))$ to (1.1) on $[0,T]$. We are interested in whether this solution can be continued for all time or if there exists a blow-up in finite time. A natural subsequent step is to prove a continuation argument which gives necessary and sufficient conditions for global existence. Recall that we used the Picard Theorem on a Banach Space to prove local existence, for fixed ϵ, to the regularized system (3.1) in *Lemma 3.1*. This theorem has a natural continuation argument. The family of solutions can be extended in time provided $|1/A^\epsilon|_\infty$, $\|A^\epsilon\|_m$, and $\|\rho^\epsilon\|_m$ remain bounded [31]. This argument does not directly apply to the solution of the original system and to prove a similar result we need the following theorem.

Theorem 5.1 (Continuity in the High Norms). *Given initial conditions $(A_0(x), \rho_0(x)) \in V^m$, for $m > 3$, which satisfy the conditions stated in Theorem 2.1. Let $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon > 0}$ be the family of solutions to (3.1) and (A, ρ) be the solution described in Theorem 2.1. The following hold:*

(1) $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$ and (A, ρ) are uniformly bounded in $C^{weak}([0,T]; V^m)$.

(2) $(A,\rho)\in C([0,T];V^m)\cap C^1([0,T];V^{m-2})$.

Proof. From *Lemma 4.1* we conclude that:

$$\sup_{0\le t\le T}\|(A^\epsilon,\rho^\epsilon)\|_{V^m}\le K. \tag{5.1}$$

Furthermore, automatically from (1.1):

$$\sup_{0\le t\le T}\left\|\frac{\partial}{\partial t}(A^\epsilon,\rho^\epsilon)\right\|_{V^{m-2}}\le\widetilde{K}. \tag{5.2}$$

We need to show that the limiting solution is continuous in the weak topology of $V^m(\Omega)$. From **Definition 1** in Section 2 it suffices to show that $(A,\phi_1)_m$ and $(\rho,\phi_2)_m$, where these inner-products are defined by (2.1), are continuous scalar functions $\forall\ \phi_1,\phi_2\in H^m$. Actually, since H^{-m} is the dual of H^m we simply need to prove that for all $\psi\in H^{-m}$ the following is true: $(\psi,A^\epsilon)_{L^2}\rightharpoonup(\psi,A)_{L^2}$. The same needs to hold for ρ. Previously we proved that $A^\epsilon\to A$ in the intermediate norms, *i.e.* in $C([0,T);H^{m'})$, where $m'<m$. This implies that $A^\epsilon\rightharpoonup A$. Consider the L^2-inner product of $\psi\in H^{-m}$ and $A^\epsilon-A$:

$$(\psi,A^\epsilon-A)_{L^2}=(\psi-\phi_j,A^\epsilon)_{L^2}+(\phi_j,A^\epsilon-A)_{L^2}+(\phi_j-\psi,A)_{L^2}, \tag{5.3}$$

where $\{\phi_j\}_{j\in\mathbb{N}}$ is a sequence in $H^{-m'}$ which converges strongly in H^{-m} to ψ. Such a sequence exists because $H^{-m'}$ is dense in H^{-m}. These terms are bounded above on $[0,T]$:

$$\begin{aligned}(\psi-\phi_j,A^\epsilon)_{L^2}&\le\|\psi-\phi_j\|_{-m}\|A^\epsilon\|_m\le K\delta/3,\\(\phi_j,A^\epsilon-A)_{L^2}&\le\|\phi_j\|_{-m'}\|A^\epsilon-A\|_{m'}\le K_2\delta/3,\\(\phi_j-\psi,A)_{L^2}&\le\|\psi-\phi_j\|_{-m}\|A\|_m\le K\delta/3.\end{aligned}$$

These inequalities substituted into (5.3) gives that $(\psi,A^\epsilon-A)_{L^2}\to 0$. The same argument can be made for ρ and this wraps up the proof of part *1*.

We are left to prove that $(A,\rho)\in C([0,T];V^m(\Omega))\cap C^1([0,T];V^{m-2}(\Omega))$. Thanks to part *1* it suffices to show that $\|A(t)\|_m$ and $\|\rho(t)\|_m$ are continuous functions in time. We take advantage of (4.6) by integrating it on the interval $[0,T]$:

$$E_m(T)+\delta_1\int_0^T\|J_\epsilon\nabla\rho^\epsilon\|_m^2\,dt+\delta_2\int_0^T\|J_\epsilon\nabla A^\epsilon\|_m^2\,dt\lesssim E_m(0)+\int_0^T\{CE_m(t)+D_0\}\,dt.$$

Applying Grönwall's Lemma we obtain that $E_m(T)\le(E_m(0)-D_0/C)e^{CT}+D_0/C$. Taking the limit as $T\to 0^+$ we see that $E_m(t)$ is continuous at $t=0^+$. Furthermore, being that $E_m(t)$ is bounded on $[0,T]$ and $\delta_1,\delta_2>0$ the inequality above implies that $(A,\rho)\in L^2([0,T];V^{m+1}(\Omega))$. Thus, for *a.e* $t_0\in[0,T]$ then $(A(t_0),\rho(t_0))\in V^{m+1}$. Indeed, the initial conditions have gained regularity. Take an arbitrarily small t_0 and let $(A(t_0),\rho(t_0))$ to be a new set of initial conditions. Running through the same existence and uniqueness arguments we obtain a solution (A,ρ) which exist on an interval $[t_0, T_1]$, $(A, \rho) \in C([t_0, T_1]; V^{m'})$, where now $m' < m+1$. In view of the fact that for $m > 3$, E_m and E_{m+1} satisfy the same differential inequality then $T_1 \geq T$. Uniqueness and the arbitrary choice of t_0 implies that $(A, \rho) \in C([0, T]; V^m)$. Furthermore, by virtue of the equation then $(A, \rho) \in C^1([0, T]; V^{m-2})$. □

Remark 5.1. From (4.6) we know we have control of the V^{m+1} norm as long as we have control $|A(t_0)|_\infty$, $|\rho(t_0)|_\infty$, $|\nabla A(t_0)|_\infty$, $|\nabla \rho(t_0)|_\infty$ and M. Furthermore, control of $|1/A(t_0)|_\infty$ implies control of M.

Fortunately, we find that the terms mentioned in *Remark 5.1* are interdependent and we can obtain a dominating term. However, before we discuss this we state and prove a regularity argument.

Theorem 5.2 (Regularity). *The solutions A, ρ of the system* (1.1) *obtained from Theorem 2.1 are in the space $C^\infty((0, T) \times \Omega)$.*

Proof. Since $(A, \rho) \in C([0, T]; V^m) \cap C^1([0, T]; V^{m-2})$ from Sobolev embedding estimates $(A, \rho) \in C([0, T]; C^{m-s}) \cap C^1([0, T]; C^{m-2-s})$ for $s > 1$. This will give us smoothness in space. To obtain smoothness in time we simply look at the time-derivates of the system of equations (1.1) and use a bootstrap argument. □

Next we show that if the appropriate initial and boundary data are chosen for A then only control of $|\nabla \rho(t_0)|_\infty$ is needed to continue the solution. We prove this in the following sequence of lemmas. The first one states that $|\nabla \rho|_\infty$ and $|\nabla A|_\infty$ controls $|\rho|_\infty$ and $|A|_\infty$ respectively. This holds because there is a bound for the mass of ρ and A on any finite time interval.

Lemma 5.1. *Let A and ρ be solutions from Theorem 2.1 with initial conditions $A_0(x)$ and $\rho_0(x)$, for $1 \leq p \leq \infty$ the following estimate holds for A and ρ on $[0, T]$ for any $T > 0$:*

$$\|u(\cdot, t)\|_{L^p} \leq c \|\nabla u\|_{L^p} + (\overline{B} + A^o)T, \tag{5.4}$$

for all $t \in [0, T]$.

Proof. Adding both equations in the system (1.1) we obtain that $\int \rho(x, t) dx \leq \left(\overline{B} + A^o\right) t$. The same estimate holds for A. Since Ω is the unit torus the average value of a function u is given by $\overline{u} = \int u dx$. Now, by Poincaré inequality $\|u\|_{L^p} \leq c \|\nabla u\|_{L^p} + \|\overline{u}\|_{L^p}$. This gives the final result. □

Furthermore, since there is a max principle for the attractiveness value equation we prove that if $A(x, 0) > A^o \neq 0$ for all x then $A(x, t) \geq A^o$ during the interval of existence. We state this result formally in following lemma.

Lemma 5.2 (Lower-Bound of Attractiveness Value). *Let $\Omega = \mathbb{T}^2$ and A, $\rho \in$*

$C([0,T];C^2(\Omega)) \cap C^1([0,T];C(\Omega))$ *be a solutions to* (1.1) *with initial conditions:*

$$A(x,0) = A_0(x) > A^o,$$
$$\rho(x,0) = \rho_0(x).$$

Then $A(x,t) \geq A^o$ *in* Ω *for all* $t \in [0,T]$.

Proof. We see directly from (1.1) that $A, \rho \geq 0$. Let $w = A^o - A$ then w satisfies: $w_t = \eta\Delta w - w + w\rho - A^o\rho$. Since both ρ and A^o are nonnegative then we have:

$$w_t - \eta\Delta w - \lambda w \leq 0, \tag{5.5}$$

where $\lambda = \sup_{0\leq t\leq T} |\rho(\cdot,t) - 1|_\infty$. Then $w = e^{\lambda t}v$ satisfies (5.5) if v satisfies $v_t - \eta\Delta v \leq 0$. From the initial data we know that $w(x,0) < 0$ for all $x \in \Omega$ and the same is true for v. By continuity in time v must remain nonnegative for some nontrivial time interval say $0 < t < t_0$. Assume that at t_0 we have that $v(x_0,t_0) = 0$ for some x_0. This means that $v_t(x_0,t_0) \geq 0$ and since we have a maximum then $-\Delta v(x_0,t_0) \geq 0$ which is a contradiction unless $v(x,t_0) = 0$ for all $x \in \Omega$. Therefore, $v(x,t) \leq 0$ and since w and v have the same sign then $w(x,t) \leq 0$. This proves the result. □

Lemma 5.2 tells us that if $|\rho|_\infty$ is bounded then $A > A^o$, provided we have appropriate initial and boundary data. We also need for the solutions to the regularized model to remain bounded from below. However, we know that this is true on $[0,T]$ as was discussed earlier. In addition, we prove that if $|\nabla\rho|_\infty$ remains bounded then $|\nabla A|_\infty$ also remains bounded. This will be demonstrated in the following two lemmas.

Lemma 5.3. *Let* (A,ρ) *satisfy* (1.1) *in the classical sense and assume that* $|\nabla\rho|_\infty$ *is bounded on* $[0,T]$, *for* $T > 0$ *then*

$$\|\nabla A(\cdot,t)\|_{L^2}^2 \leq \left(\|\nabla A(\cdot,0)\|_{L^2}^2 - \tilde{C}\right) e^{C(\eta,|\nabla\rho|_\infty,A^o,\overline{B})T} + \tilde{C}, \tag{5.6}$$

where $\tilde{C} = \tilde{C}(\eta, |\nabla\rho|_\infty, A^o, \overline{B})$. *This holds* $\forall t \in [0,T]$.

Proof.

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\nabla A|^2\,dx &= \int \nabla A \cdot \nabla A_t dx \\
&\overset{(1.1)}{=} \int \nabla A \cdot \nabla\left(\eta\Delta A - A + A\rho + A^o\right)dx \\
&= -\eta\int |\Delta A|^2\,dx - \int |\nabla A|^2\,dx + \int \nabla A \cdot \nabla(A\rho)dx \\
&= -\eta\int |\Delta A|^2\,dx - \int |\nabla A|^2\,dx + \int |\nabla A|^2\rho dx + \int A\nabla A \cdot \nabla\rho dx \\
&\overset{\text{Cauchy-Schwarz}}{\leq} -\eta\|\Delta A\|_{L^2}^2 + (|\rho|_\infty - 1)\|\nabla A\|_{L^2}^2 + |\nabla\rho|_\infty\left(\|A\|_{L^2}^2 + \|\nabla A\|_{L^2}^2\right) \\
&\overset{(5.4)}{\leq} C(\eta, |\rho|_\infty, |\nabla\rho|_\infty)\|\nabla A\|_{L^2}^2 + C(|\nabla\rho|_\infty, A^o, \overline{B}, T).
\end{aligned}$$

Integrating this and using (5.4) for $p = \infty$ gives the desired result (5.6). □

Lemma 5.4. *Let (A, ρ) satisfy (1.1) in the classical sense and assume that $|\nabla\rho|_\infty$ is bounded on $[0, T]$, for $T > 0$ then*

$$|\nabla A(\cdot, t)|_\infty \leq c_4 \max\{|\nabla A(\cdot, 0)|_\infty, \left(\|\nabla A(\cdot, 0)\|^2_{L^2} - \tilde{C}\right) e^{CT} + \tilde{C}\} \tag{5.7}$$

$\forall t \in [0, T]$. The constants C and $\tilde{C}$ are defined as in Lemma 5.3.

The proof of *Lemma 5.4* uses the Moser-Alikakos iteration.[1]

Proof. Let $s \geq 2$:

$$\begin{aligned}
\frac{1}{s}\frac{d}{dt}\int |\nabla A|^s\, dx &= \int |\nabla A|^{s-1} \nabla A_t dx \\
\overset{(1.1)}{=} &\int |\nabla A|^{s-1} \nabla\left(\eta\Delta A - A + A\rho + A^o\right) dx \\
&\leq -\eta(s-1)\int |\nabla A|^{s-2}|\Delta A|^2 dx - \int |\nabla A|^s dx + \int |\nabla A|^s \rho dx \\
&\quad + \int A|\nabla A|^{s-1}|\nabla\rho|\, dx \\
\overset{\text{Hölder's Ineq.}}{\leq} &-\frac{4\eta(s-1)}{s^2}\left\|\nabla\left(|\nabla A|^{s/2}\right)\right\|^2_{L^2} + |\nabla\rho|_\infty \|\nabla A\|^{s-1}_{L^s}\|A\|_{L^2} \\
&\quad + (|\rho|_\infty - 1)\|\nabla A\|^s_{L^s} \\
\overset{(5.4)}{=} &-\frac{4\eta(s-1)}{s^2}\left\|\nabla\left(|\nabla A|^{s/2}\right)\right\|^2_{L^2} + c_1\|\nabla A\|^s_{L^s} + c_2,
\end{aligned}$$

where $c_1 = c_1(|\nabla\rho|_\infty, |\rho|_\infty, A^o, \overline{B})$ and $c_2 = c_2(A^o, \overline{B})$. Multiplying both sides by s, $s \geq 2$ gives:

$$\frac{d}{dt}\int |\nabla A|^s dx \leq -2\eta\left\|\nabla\left(|\nabla A|^{s/2}\right)\right\|^2_{L^2} + sc_1\|\nabla A\|^s_{L^s} + sc_2.$$

We need to make use of an extended Sobolev inequality:[17]

$$-\|\nabla u\|^2_{L^2} \leq -\frac{(1-\epsilon)}{\epsilon}\|u\|^2_{L^2} + \frac{\overline{c}}{\epsilon^2}\|u\|^2_{L^1}. \tag{5.8}$$

A derivation of (5.8) can be found in Appendix A.4. Taking $u = |\nabla A|^{s/2}$ gives:

$$\frac{d}{dt}\int |\nabla A|^s dx \leq -\frac{2\eta(1-\epsilon)}{\epsilon}\|\nabla A\|^s_s + \frac{c_0}{\epsilon^2}\|\nabla A\|^s_{L^{s/2}} + c_1 s\|\nabla A\|^s_{L^s} + sc_2,$$

Choose $\epsilon = \frac{\eta}{sc_1+\eta}$ noting that $s > \eta \in [0, 1]$ (refer to [34]) then:

$$\frac{d}{dt}\int |\nabla A|^s_{L^s} dx \leq -c_1 s\|\nabla A\|^s_{L^s} + c_3 s^2\|\nabla A\|^s_{L^{s/2}} + sc_2.$$

By multiplying both sides by $e^{c_1 st}$ the above inequality is equivalent to

$$\frac{d}{dt}\left\{e^{c_1 st}\|\nabla A\|^s\right\} \leq e^{c_1 st}\left(c_3 s^2\|\nabla A\|^2_{L^{s/2}} + c_2 s\right).$$

Integrating this over $[0,t]$ gives:

$$\begin{aligned}
e^{c_1 st}\left\|\nabla A(\cdot,t)\right\|_{L^s}^s &\leq \left\|\nabla A(\cdot,0)\right\|_{L^s}^s + \sup_{0\leq\tau\leq t}\left\|\nabla A(\cdot,\tau)\right\|_{L^{s/2}}^s \int_0^t c_3 s^2 e^{c_1 s\tau} d\tau + \int_0^t c_2 e^{c_1 s\tau} s d\tau \\
&\leq \left\|\nabla A(\cdot,0)\right\|_{L^s}^s + \sup_{0\leq\tau\leq t}\left\|\nabla A(\cdot,\tau)\right\|_{L^{s/2}}^s c_4 s(e^{c_1 st}-1) + c_5(e^{c_1 st}-1).
\end{aligned}$$

Therefore,

$$\left\|\nabla A(\cdot,t)\right\|_{L^s}^s \leq \left(\left|\nabla A(\cdot,0)\right|_\infty + c_6\right)^s + c_4 s \sup_{0\leq\tau\leq t}\left\|\nabla A(\cdot,t)\right\|_{L^{s/2}}^s, \tag{5.9}$$

where
$c_6 = \max\{1, c_5\}$. Define $M(s) = \max\{\left|\nabla A(\cdot,0)\right|_\infty + c_6, \sup_{0\leq t\leq T}\left\|\nabla A(\cdot,t)\right\|_{L^s}\}$. From (5.9) we conclude that

$$M(s) \leq (c_7 s)^{1/s} M(s/2). \tag{5.10}$$

Let $s = 2^k$ for $k \in \mathbb{N}$ the recursive relation (5.10) gives:

$$M(2^k) \leq (c_7)^{\sum_{j=1}^k 2^{-j}} (2)^{\sum_{j=1}^k j2^{-j}} M(1).$$

Since both sums $\sum_{j=1}^k 2^{-j}$ and $\sum_{j=1}^k j2^{-j}$ converge as $k\to\infty$ taking the limit as $s\to\infty$ we get:

$$\begin{aligned}
\left|\nabla A(\cdot,T)\right|_\infty &\leq \lim_{s\to\infty} M(s) \\
&\leq (c_7)^{\sum_{j=1}^\infty 2^{-j}} (2)^{\sum_{j=1}^\infty j2^{-j}} M(1),
\end{aligned}$$

Applying *Lemma 5.3* gives the final result. □

From *Theorem 5.1* and *Lemma 5.2-5.4* proved above we obtain necessary and sufficient conditions for the continuation of the solution to (1.1).

Corollary 5.1. *Given initial conditions* $(A(x,0),\rho(x,0)) \in V^m$, $m \geq 4$ *such that* $A(x,0) > A^o$ *and 'no-flux' boundary conditions, there exist a maximal time of existence* $0 < T_{max} \leq \infty$ *and a unique solution* $(A(x,t),\rho(x,t)) \in C([0,T_{max});V^m) \cap C^1([0,T_{max});V^{m-2})$ *the the system* (1.1). *Furthermore, if* T_{max} *is finite then* $\lim_{t\to T_{max}}\left|\nabla\rho\right|_\infty = \infty$.

6. Analysis of a modified PDE model of criminal behavior and its relation to Keller-Segel Model

Though we succeeded in proving local existence and uniqueness of solutions to (1.1) the question of whether the solutions can be extended for all time has not been addressed. To be confident that we have a robust model, suitable for the target application, we need insight on global existence and/or possible blow up. Working with a strongly coupled system of nonlinear PDEs makes it difficult to apply the usual techniques to prove well-posedness. Fortunately, as was mentioned before, there is an evident relation between the model for residential burglaries and the Keller-Segel model for chemotaxis, developed in [28] by Keller and Segel in 1971. This is not surprising since both processes are usually modeled by a parabolic-parabolic system and include motion up gradients of some external field. Chemotaxis is the influence of a chemical substance in the environment on the movement of a mobile species. This process is key in cellular communications. Keller and Segel developed a general model for the chemotaxis phase of aggregation of slime mold, *i.e* Dictyostelium Discoidium in [28]. There has been a great deal of analysis on various versions of the Keller-Segel model since it was developed and research is still in progress [7,10,13,15,21, 29,11]. Thus far the most studied version is:

$$\frac{\partial u}{\partial t} = \kappa \Delta u - \chi \nabla \cdot (u \nabla v), \tag{6.1a}$$

$$\epsilon \frac{\partial v}{\partial t} = k_c \Delta v - \alpha v + \beta u. \tag{6.1b}$$

with Neumann boundary conditions. In (6.1) u is the myxamoebae density of slime mold and v the chemo-attractant concentration. Comparing this model to (1.1) we can see that the chemo-attractant density is comparable to the attractiveness value. It is worth noting that chemotaxis is sometimes modeled by an elliptic-parabolic system; however, in the residential burglaries model the timescale of the change in attractiveness value is similar to the change in criminal density. From (6.1) we see that the myxamoebae move up gradients of chemo-attract concentration like criminals move up gradients of attractiveness value. Global existence and finite time blow-up of the (6.1) is highly dependent on the dimension. In one-dimension finite time blow up cannot occur [8]. In two-dimensions it has been shown, by Corrias and Calvez [6], that the solution exist globally in time if the initial mass is below the critical quantity 8π. If the initial mass is above 8π then aggregation occurs in the case when $\epsilon = 0$[21,20]. As far as we know the blow-up results for the fully parabolic system has not been proved. For higher dimensions, d, there exists a similar critical quantity that is governed by the $L^{d/2}$-norm of the initial myxamoebae density. Although the most studied version of the Keller-Segel model is (6.1) various variations of the model have also been analyzed. A comprehensive summary of much of this work can be found in [23] and [24]. In a sense, the model given by (1.1) can be thought of as a generalized and more complicated version of (6.1), which includes growth and decay of the myxamoebae density and the chemo-attractant. We want to take advantage the extensive body of work done on (6.1) as a first step to obtaining insight on the global existence or finite time blow-up of (1.1). To accomplish this analyze a simplified model of (1.1). This will ease the mathematical analysis while maintaining fundamental assumptions made in [34]. The model we propose is:

$$\epsilon \frac{\partial A}{\partial t} = \eta \Delta A - A + \beta \rho + A^o(x), \tag{6.2a}$$

$$\frac{\partial \rho}{\partial t} = \Delta \rho - 2\nabla \cdot (\rho \nabla A) + \overline{B}(x) - f(A)\rho. \tag{6.2b}$$

From now on we work in all of $\mathbb{R}^2$. Notice that now A^o and $\overline{B}$ are functions of the space variable and must have sufficient decay as $|x| \to \infty$. Model (6.2) makes three simplifications to (1.1). First, the advection speed is now given simply by $|\nabla A|$. The second modification is that the attractiveness value increases with the number of criminals with constant of proportionality β, *i.e* we replace $A\rho$ with $\beta\rho$ in (1.1a). We have no reason to believe that this modification will decrease the accuracy of the model. Finally, the criminal density decays with a rate of $f(A)$ and we assume that $f(A)$ has a lower and upper bound.

6.1. *Useful Properties of the Modified Residential Burglaries Model*

It is not surprising that (6.1) is the most studied version of the Keller-Segel model since is possesses properties that facilitates mathematical anlysis. There are three properties worth noting. First, the system (6.1) conserves mass of the cell density. Furthermore, one can express the chemo-attractant concentration as the convolution of the Bessel Kernel, $\mathcal{B}_\eta(z) = \frac{1}{4\eta\pi}\int_0^\infty \frac{1}{t}e^{-\frac{|z|^2}{4\eta t}-t}dt$, and the cell density. In two-dimensions this is especially useful for proving blow-up results given large enough initial mass of the cell density. Most importantly, this model, after non-dimensionalization, has a Lyapunov functional [6]:

$$\mathcal{F}(t) = \int u\log(u)dx - \int uv\, dx + \frac{1}{2}\int |\nabla v|^2\, dx + \int \alpha v^2 dx.$$

This functional is key in proving global existence. The model (6.2) does not possess these exact properties; however, it does possess ones which are useful enough. For $v \in L^1(\Omega)$ let $M_v(t) = \int_\Omega v(x,t)dx$. As an example of a useful property if $A,\ \rho$ are solutions to (6.2) then $M_\rho(t)$ is bounded above and below. Define $f_{min} = \min_{A\in\mathbb{R}^+} f(A)$ then:

$$M_\rho(t) \le e^{-f_{min}t}\left(M_\rho(0) - \frac{M_{\overline{B}}}{f_{min}}\right) + \frac{M_{\overline{B}}}{f_{min}}. \tag{6.3}$$

Replacing f_{min} with f_{max} gives a similar lower bound for $M_\rho(t)$. Another key property is the explicit expression of the attractiveness value in terms of the criminal density in the quasi-static case, *i.e* $\epsilon = 0$:

$$A(x) = \mathcal{B}_\eta * (\beta\rho + A^o) \tag{6.4}$$

We conjecture that solutions to (6.2) satisfy an energy functional whose upper bound can be controlled with time. Being that this is beyond the scope of this paper we only mention that proving such an energy functional is important for proving global existence via the Lyapunov functional method discussed in the introduction.

6.2. *Blow-up of a Modified Residential Burglaries Model*

In this section we explore the possibility of blow-up in finite time of the solution to the modified residential burglaries model (6.2) in the case where $\epsilon = 0$. It turns out, that similar to the Keller-Segel model, if the lower-bound on the mass of the criminal density is large enough there is mass concentration on a set of measure zero. Let $M_\rho^{min} = \min\{M_\rho(0), M_{\overline{B}}/f_{max}\}$ and $M_\rho^{max} = \max\{M_\rho(0), M_{\overline{B}}/f_{min}\}$ and for a function v we denote the finite second moment by $I_v = \int |x|^2 v dx$. We state this blow-up result in the following theorem.

Theorem 6.1 (Blow-up of a Modified Residential Burglaries Model). *Let* $(A(x,0), \rho(x,0)) \in L^1(\mathbb{R}^2)$ *be initial data such that* $\left(\beta M_\rho^{min} - 4\pi\right) M_\rho^{min} > \pi I_{\overline{B}}$. *Furthermore, let* ρ *be the non-negative smooth solution to* (6.2b) *and that* A *has reached a steady state and is defined by* (6.4)*, then* A, ρ *are a non-negative smooth solutions to* (6.2) *(when* $\epsilon = 0$*). Then, if the initial second moment is small enough. That is if*

$$\int |x|^2 \rho dx \leq \frac{1}{K^2}\left[\left(\frac{\beta}{\pi} M_\rho^{min} - 4\right) M_\rho^{min} - I_{\overline{B}}\right]^2, \tag{6.5}$$

where $I_{\overline{B}}(x) = \int \overline{B} dx$, $K = \left[\frac{2\beta}{\pi} C \left(M_\rho^{max}\right)^{3/2} + a_1 \left(M_\rho^{max}\right)^{1/2}\right]$, $a_1 = 4\left\|\nabla \mathcal{B}_\eta(x)\right\|_1 |A^o(x)|_\infty$ *and* C *a constant, then there exists a finite time singularity.*

Proof. Consider the time evolution of the second moment of ρ, $I(t) = \int_{\mathbb{R}^2} |x|^2 \rho dx$:

$$\begin{aligned}
\frac{dI}{dt} &= \int_{\mathbb{R}^2} |x|^2 \left(\Delta\rho - 2\nabla\cdot(\rho\nabla A) - f(A)\rho + \overline{B}\right) dx \\
&\leq 4\int_{\mathbb{R}^2} \rho\, dx + 4\beta\int_{\mathbb{R}^2} \rho\left(\mathbf{x}\cdot\nabla\mathcal{B}_\eta * \rho\right) dx + 4\int_{\mathbb{R}^2} \rho\left(\mathbf{x}\cdot\nabla\mathcal{B}_\eta * A^o(x)\right) dx + I_{\overline{B}}.
\end{aligned} \tag{6.6}$$

The third term of on the right in the above inequality can be bounded above using Cauchy-Schwarz Inequality and Young's inequality for convolutions [39]:

$$\begin{aligned}
4\int_{\mathbb{R}^2} \rho\,|\mathbf{x}|\,|\nabla\mathcal{B}_\eta * A^o(x)|\, dx &\leq 4\,|\nabla\mathcal{B}_\eta * A^o(x)|_\infty \int \rho\,|\mathbf{x}|\, dx. \\
&\leq \underbrace{4\left\|\nabla\mathcal{B}_\eta\right\|_{L^1} |A^o(x)|_\infty}_{a_1} M_\rho^{1/2}(t)\sqrt{I(t)}
\end{aligned} \tag{6.7}$$

We use the explicit expression of the gradient of the Bessel Kernel, $\nabla\mathcal{B}_\eta(z) = -\frac{1}{2\pi}\frac{z}{|z|^2}\int_0^\infty e^{-s-\frac{|z|^2}{4\eta s}} ds$, to bound the second term. Let $g_\eta(z) = \int_0^\infty e^{-s-\frac{|z|^2}{4\eta s}} ds$ and $d\mathcal{A} = dxdy$ then:

$$\begin{aligned}
4\beta\int_{\mathbb{R}^2} \rho\left(\mathbf{x}\cdot\nabla\mathcal{B}_\eta * \rho\right) dx &\leq -\frac{2\beta}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(x,t)\mathbf{x}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|x-y|^2} g_\eta(x-y)\rho(y,t) dy dx \\
&\leq \frac{\beta}{\pi}\iint \rho(x,t)[1 - g_\eta(x-y)]\,\rho(y,t) d\mathcal{A} - \frac{\beta}{\pi}\iint \rho(x,t)\rho(y,t) d\mathcal{A} \\
&= -\frac{\beta}{\pi} M_\rho^2(t) + \frac{\beta}{\pi}\iint \rho(x,t)\left[1 - g_\eta(x-y)\right]\rho(y,t) d\mathcal{A}.
\end{aligned}$$

Observe that $g_\eta(z)$ is a positive, radially symmetric, decreasing function with maximum of one. This implies that $0 \leq (1 - g_\eta(z)) \leq 1$. Now, consider the derivative of $(1 - g_\eta(r))$ with respect to $r = |z|$:

$$\begin{aligned}\frac{d}{dr}(1 - g_\eta(r)) &\leq \frac{r}{2\eta}\int_0^\infty \frac{1}{s}\, e^{-s-\frac{r^2}{4\eta s}}ds \\ &\leq \frac{2\pi}{\eta} r\mathcal{B}_1(\frac{r}{\sqrt{\eta}}) \\ &\leq \frac{2\pi}{\sqrt{\eta}} \sup_{\tilde{r}\in(0,1)} (\tilde{r}\mathcal{B}_1(\tilde{r})),\end{aligned}$$

where, $\tilde{r} = \frac{r}{\sqrt{\eta}}$ for $0 \leq r \leq \sqrt{\eta}$. If $C = \frac{2\pi}{\sqrt{\eta}} \max(\sup_{\tilde{r}\in(0,1)} \{\tilde{r}\mathcal{B}_1(\tilde{r})\}, 1)$ then $(1 - g_\eta(z)) \leq C\,|z|$. Hence, we have:

$$\begin{aligned}4\beta\int_{\mathbb{R}^2} \rho\,(\mathbf{x}\cdot\nabla\mathcal{B}_\eta * \rho)\,dx &\leq -\frac{\beta}{\pi}M_\rho^2(t) + \frac{2\beta}{\pi}CM_\rho(t)\int_{\mathbb{R}^2}\rho(x,t)\,|x|\,dx \\ \text{C.S.}\ &\leq -\frac{\beta}{\pi}M_\rho^2(t) + \frac{2\beta}{\pi}C\,(M_\rho(t))^{3/2}\sqrt{I(t)}. \qquad (6.8)\end{aligned}$$

Substituting (6.7) and (6.8) into (6.6) gives:

$$\begin{aligned}\frac{dI}{dt} &\leq \left(4 - \frac{\beta}{\pi}M_\rho(t)\right)M_\rho(t) + \frac{2\beta}{\pi}CM_\rho^{3/2}(t)\sqrt{I(t)} + a_1M_\rho^{1/2}(t)\sqrt{I(t)} + I_{\overline{B}} \\ &\leq \left(4 - \frac{\beta}{\pi}M_\rho(t)\right)M_\rho(t) + \left(\frac{2\beta}{\pi}CM_\rho^{3/2}(t) + a_1M_\rho^{1/2}(t)\right)\sqrt{I(t)} + I_{\overline{B}} \\ &\leq \left(4 - \frac{\beta}{\pi}M_\rho^{min}\right)M_\rho^{min} + \underbrace{\left(\frac{2\beta}{\pi}C\left(M_\rho^{max}\right)^{3/2} + a_1\left(M_\rho^{max}\right)^{1/2}\right)}_{K}\sqrt{I(t)} + I_{\overline{B}}.\end{aligned}$$

In the last inequality we use the fact that the initial conditions are chosen so that $\beta M_\rho^{\min} > 4\pi$. Integrating on $[0, t)$ gives the integral inequality:

$$I(t) \leq I(0) + \int_0^t g(I(s))ds, \qquad (6.9)$$

where $g(I(t)) = \left(4 - \frac{\beta}{\pi}M_\rho^{min}\right)M_\rho^{min} + K\sqrt{I(t)} + I_{\overline{B}}$. The function $g(I)$ is continuous, increasing and such that $g(I(t^*)) = 0$ for $t^* > 0$ such that:

$$I(t^*) = \frac{1}{K^2}\left[\left(\frac{\beta}{\pi}M_\rho^{min} - 4\right)M_\rho^{min} - I_{\overline{B}}\right]^2,$$

where K is defined in the theorem. Since $I(0) \leq I(t^*)$ by continuity of g there exists a $\tilde{t} > 0$ such that $\int_0^{\tilde{t}} g(I(s))ds < 0$. Hence, $I(\tilde{t}) < I(0)$. Repeating this process will eventually give that $I(t) = 0$ for some positive t which proves the result. □

From *Theorem 6.1* we conjecture that a logarithmic sensitivity function is more suitable than a linear sensitivity function. Moreover, from the maximum principle of the attractiveness value a lower bound on $f(A)$ is implicit in the original model. Hence, setting $f(A) = A$ is only eliminating the upper bound on $f(A)$. This would only help prevent blow-up. The remaining difference between the two models is less obvious to analyze. We conjecture that the nonlinear $A\rho$ aids blow-up more than $\beta\rho$. This is because we expect, and indeed we observe numerically, that A and ρ grow and decay together. Hence, we have that $A\rho \approx \rho^2$ which would aid blow-up more so than $\beta\rho$ would.

6.3. *Exploring Blow-up of a Modified Residential Burglaries Model in 1D*

Although we see blow-up in the modified model for large enough mass of the initial criminal density in two dimensions, a similar type of blow-up in finite time of the model (6.2) cannot occur in one dimension. This is due to change of properties of the Bessel Kernel in one dimension. In fact, a simple computation shows that the second moment will always be bounded below by something positive. For simplicity of notation we take $\eta = 1$, in this case in one-dimension we have that $\mathcal{B}(x) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1}{t^{1/2}} e^{-\frac{|x|^2}{4t} - t} dt$ and $\partial_x \mathcal{B}(x) = -\frac{x}{\sqrt{\pi}}\int_0^\infty \frac{1}{4t^{3/2}} e^{-\frac{|x|^2}{4t} - t} dt = -\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac{|x|^2}{4s^2} - s^2} ds$. In contrast to the previous section we now seek a bound from below for the second moment.

$$
\begin{aligned}
\frac{dI}{dt} &= 2M_\rho(t) + 4\beta\int_{\mathbb{R}^2} \rho\left(x\partial_x\mathcal{B} * \rho\right)dx + 4\int_{\mathbb{R}^2} \rho\left(x\partial_x\mathcal{B} * A^o(x)\right)dx - \int_{\mathbb{R}^2} f(A)\,|x|^2\rho dx + I_{\overline{B}} \\
&\geq 2M_\rho^{min} + I_{\overline{B}} + 4\beta\int_{\mathbb{R}^2} \rho\left(x\partial_x\mathcal{B} * \rho\right)dx + 4\int_{\mathbb{R}^2} \rho\left(x\partial_x\mathcal{B} * A^o(x)\right)dx - f_{max} I(t) \\
&\geq 2M_\rho^{min} + I_{\overline{B}} - f_{max} I(t) - \left[4\left|\mathcal{B}_x\right|_\infty \left(\beta\left(M_\rho^{max}\right)^{3/2} + \left\|A^o\right\|_1 M_\rho^{max}\right)\right] I(t)^{1/2} \\
&\geq C_1 - C_2 I(t),
\end{aligned}
$$

where, $C_1 = 2M_\rho^{min} + I_{\overline{B}} - \delta\left[4\left|\mathcal{B}_x\right|_\infty \left(\beta\left(M_\rho^{max}\right)^{3/2} + \left\|A^o\right\|_1 M_\rho^{max}\right)\right]^2$ and $C_2 = \frac{1}{\delta} - f_{max}$. We choose δ small enough such that $C_1 > 0$. This implies that $I(t) \geq e^{-C_2 t}\left(I(0) - C_1/C_2\right) + C_1/C_2$, which has a bound from below for all time. Hence, if there is blow-up in finite time we cannot show it via this method. This agrees with preliminary numerical results which show finite time blow-up in two-dimensions but not in one-dimension. We hope to address this issue further in a future article.

7. Discussion

The overarching goal of the development of model (1.1) by Short *et al.* is to help understand the spatio-temporal dynamics of residential burglary "hotspots" to aid law-enforcement in the mobilization of their resources. In this paper we studied the well-posedness of this model to determine whether it is a suitable model for the target application. In particular, we know that certain types of finite time blow-up would invalidate the model. For example, blow-up in the L^∞ norm of the criminal density would not make any physical sense. As a first step to determining whether this model is well-posed we proved local existence of classical solutions. Furthermore, we know that with no-flux boundary conditions the attractiveness value will never go below its static component A^o. From the continuation argument we know that the model has a global solution provided the L^∞ bound of the gradient of ρ remains bounded. In the case of blow-up we also know that if the L^∞ norm of the gradient of A blows up in finite time then the same has to be true for the gradient of ρ. In the final part of the paper we explored the connections of the residential burglary model and the Keller-Segel model for chemotaxis, which has been vastly studied. Considering a modified residential burglary model we determined that the logarithmic sensitivity function in (1.1) is essential to preventing blow-up. In fact, preliminary numerical results show finite time blow-up for (1.1) with $\chi(A) = A$ with no other modifications in two-dimensions. In one-dimension no such blow-up has been observed. This serves to confirm the connections between the Keller-Segel model and the residential burglary model.

Appendix A. Additional Computations

A.1. *Computations for Theorem 3.1*

Computations for F_2

$$\begin{aligned}\|F_2(v_1) - F_2(v_2)\|_2 \leq \left\|J_\epsilon^2 \Delta(\rho_1 - \rho_2)\right\|_2 + 2\left\|J_\epsilon\left[\nabla \cdot \frac{\rho_1}{A_1} J_\epsilon \nabla A_1 - \nabla \cdot \frac{\rho_2}{A_2} J_\epsilon \nabla A_2\right]\right\|_2 \\ + \|\rho_1 A_1 - A_2 \rho_2\|_2 = S_1 + S_2 + S_3.\end{aligned}$$

The terms S_1 and S_3 appeared in the inequality for F_1; therefore, we are only concerned with S_2:

$$\begin{aligned}\frac{1}{2} S_2 &\lesssim \frac{1}{\epsilon^2}\left\|\frac{\rho_1}{A_1} J_\epsilon \nabla A_1 - \frac{\rho_2}{A_2} J_\epsilon \nabla A_2\right\|_1 \\ &\leq \frac{1}{\epsilon^2}\left(\left\|\frac{\rho_1}{A_1} J_\epsilon \nabla\left(A_1 - A_2\right)\right\|_1 + \left\|J_\epsilon \nabla A_2\left(\frac{\rho_1}{A_1} - \frac{\rho_2}{A_2}\right)\right\|_1\right) \\ &\lesssim \frac{1}{\epsilon^2}\left(\left|\frac{\rho_1}{A_1}\right|_\infty \left\|D\left\{J_\epsilon \nabla\left(A_1 - A_2\right)\right\}\right\|_0 + \left|J_\epsilon \nabla\left(A_1 - A_2\right)\right|_\infty \left\|D\left(\frac{\rho_1}{A_1}\right)\right\|_0\right) \\ &\quad + \frac{1}{\epsilon^2}\left(\left|J_\epsilon \nabla A_2\right|_\infty \left\|D\left(\frac{\rho_1}{A_1} - \frac{\rho_2}{A_2}\right)\right\|_0 + \left|\frac{\rho_1}{A_1} - \frac{\rho_2}{A_2}\right|_\infty \left\|D J_\epsilon \nabla A_1\right\|_0\right) \\ &= \frac{1}{\epsilon^2}\left(R_1 + R_2 + R_3 + R_4\right).\end{aligned}$$

R_1 can be easily bounded, without any additional factors of $1/\epsilon$, by $\left|A_1^{-1}\right|_\infty |\rho_1|_\infty \|A_1 - A_2\|_2$. On the other hand, for R_2 we need to use (5) of *Lemma*

2.1 and (*1*) of *Lemma 2.2*. More precisely, we have:

$$R_2 \lesssim \frac{1}{\epsilon} \|A_1 - A_2\|_2 \left(\left|\frac{1}{A_1}\right|_\infty \|\rho_1\|_1 + \left|\frac{1}{A_1}\right|_\infty^2 \|A\|_1 |\rho_1|_\infty \right).$$

The next term requires more work, basically repeated applications of the *Lemma 2.2*.

$$\begin{aligned} R_3 &\lesssim \frac{1}{\epsilon} \|A_2\|_2 \left\| \frac{A_2\rho_1 - A_1\rho_2}{A_1A_2} \right\|_1 \\ &\lesssim \frac{1}{\epsilon} \|A_2\|_2 \left\{ \left| \frac{1}{A_1A_2} \right|_\infty \|\rho_1 A_2 - \rho_2 A_1\|_1 + |\rho_1 A_2 - \rho_2 A_1|_\infty \left\| D\left(\frac{1}{A_1A_2}\right) \right\|_0 \right\} \\ &\lesssim \frac{1}{\epsilon} \|A_2\|_2 \left\{ \left| \frac{1}{A_1A_2} \right|_\infty \|\rho_1 A_2 - \rho_2 A_1\|_1 + |\rho_1 A_2 - \rho_2 A_1|_\infty \right\}. \end{aligned}$$

Since, $|v|_\infty \lesssim \|v\|_2$ and

$$\left\| D\left(\frac{1}{A_1A_2}\right) \right\|_0 \lesssim \left|\frac{1}{A_1}\right|_\infty \left|\frac{1}{A_2}\right|_\infty^2 \|\nabla A_2\|_0 + \left|\frac{1}{A_2}\right|_\infty \left|\frac{1}{A_1}\right|_\infty^2 \|\nabla A_1\|_0,$$

we have:

$$R_3 \lesssim \frac{1}{\epsilon} \|A_2\|_2 \left(\left|\frac{1}{A_1}\right|_\infty \left|\frac{1}{A_2}\right|_\infty + \left|\frac{1}{A_1}\right|_\infty \left|\frac{1}{A_2}\right|_\infty^2 \|A_2\|_1 + \left|\frac{1}{A_2}\right|_\infty \left|\frac{1}{A_1}\right|_\infty^2 \|A_1\|_1 \right) \|\rho_1 A_2 - \rho_2 A_1\|_2.$$

Finally,

$$R_4 \le \left|\frac{1}{A_1}\right|_\infty \left|\frac{1}{A_2}\right|_\infty \|A_1\|_2 |A_2\rho_1 - \rho_2 A_1|_\infty.$$

A.2. *Computations for Higher-Order Energy Estimate Estimates*

Claim 1:

$$\sum_{|\alpha|\le m} \|D^\alpha u\|_0 \|D^\alpha (uv)\|_0 \lesssim (|\nabla u|_\infty + |u|_\infty + |v|_\infty) \|u\|_m^2 + (|\nabla u|_\infty + |u|_\infty) \|v\|_m^2.$$

Proof.

$$\begin{aligned} \sum_{|\alpha|\le m} \|D^\alpha u\|_0 \|D^\alpha (uv)\|_0 &\le \sum_{|\alpha|\le m} \|D^\alpha u\|_0 \{ \|D^\alpha (uv) - uD^\alpha v\|_0 + |u|_\infty \|D^\alpha v\|_0 \} \\ &\le \|u\|_m \left(\sum_{|\alpha|\le m} \|D^\alpha(uv) - uD^\alpha v\|_0 + |u|_\infty \|v\|_m \right) \\ &\le c\|u\|_m \left\{ |\nabla u|_\infty \left\|D^{m-1} v\right\|_0 + \|D^m u\|_0 |v|_\infty + |u|_\infty \|v\|_m \right\} \end{aligned}$$

This proves the claim. □

Lemma Appendix A.1.

$$\left\| D^m \left(\frac{1}{A^\epsilon} \right) \right\|_0 \leq \sum_{k=0}^{m-1} C_k \left| \frac{1}{A^\epsilon} \right|_\infty^{k+2} |\nabla A^\epsilon|_\infty^k \left\| D^{m-k} A^\epsilon \right\|_0 , \tag{A.1}$$

where the C_k's are constants.

Proof. Using *(1)* of *Lemma 2.2* and dropping the constants we get:

$$\begin{aligned}
\left\| D^m \left(\frac{1}{A} \right) \right\|_0 &= \left\| D^{m-1} \left(\frac{\nabla A}{A^2} \right) \right\|_0 \\
&\lesssim |\nabla A|_\infty \left\| D^{m-1} \left(\frac{1}{A^2} \right) \right\|_0 + \left| \frac{1}{A} \right|_\infty^2 \left\| D^{m-1} \nabla A \right\|_0 \\
&\lesssim |\nabla A|_\infty \left(|\nabla A|_\infty \left\| D^{m-2} \left(\frac{1}{A^3} \right) \right\|_0 + \left| \frac{1}{A} \right|_\infty^3 \left\| D^{m-2} \nabla A \right\|_0 \right) + \left| \frac{1}{A} \right|_\infty^2 \left\| D^m A \right\|_0 \\
&\vdots \\
&\lesssim |\nabla A|_\infty^{m-1} \left\| D^1 \left(\frac{1}{A^m} \right) \right\|_\infty + \sum_{k=0}^{m-2} \left| \frac{1}{A} \right|_\infty^{k+2} |\nabla A|_\infty^k \left\| D^{m-k} A \right\|_0 \\
&\lesssim \sum_{k=0}^{m-1} \left| \frac{1}{A} \right|_\infty^{k+2} |\nabla A|_\infty^k \left\| D^{m-k} A \right\|_0 .
\end{aligned}$$

□

Proof. (**Lemma 4.2**)

$$\begin{aligned}
\overbrace{\int D^\alpha (J_\epsilon \nabla \rho) \cdot D^\alpha \left(\frac{\rho}{A} J^\epsilon \nabla A \right) dx}^{I_\alpha} &\leq \| D^\alpha (J_\epsilon \nabla \rho) \|_0 \left\| D^\alpha \left(\frac{\rho}{A} J^\epsilon \nabla A \right) \right\|_0 \\
&\leq \| D^\alpha (J_\epsilon \nabla \rho) \|_0 \left\| D^\alpha \left(\frac{\rho}{A} J^\epsilon \nabla A \right) - \frac{\rho}{A} D^\alpha (J_\epsilon \nabla A) \right\|_0 \\
&\quad + \| D^\alpha (J_\epsilon \nabla \rho) \|_0 \left| \frac{\rho}{A} \right|_\infty \| D^\alpha J_\epsilon \nabla A \|_0 .
\end{aligned}$$

Summing over $|\alpha| \leq m$ gives:

$$\begin{aligned}
\sum_{|\alpha| \leq m} I_\alpha &\leq \| J_\epsilon \nabla \rho \|_m \left(\sum_{|\alpha| \leq m} \left\| D^\alpha \left(\frac{\rho}{A} J^\epsilon \nabla A \right) - \frac{\rho}{A} D^\alpha (J_\epsilon \nabla A) \right\|_0 + \left| \frac{\rho}{A} \right|_\infty \| J_\epsilon \nabla A \|_m \right) \\
&\lesssim \| J_\epsilon \nabla \rho \|_m \left(\left| \nabla \left(\frac{\rho}{A} \right) \right|_\infty \left\| D^{m-1} J_\epsilon \nabla A \right\|_0 + |J_\epsilon \nabla A|_\infty \left\| D^m \frac{\rho}{A} \right\|_0 + \left| \frac{\rho}{A} \right|_\infty \| J_\epsilon \nabla A \|_m \right) .
\end{aligned}$$

We bound the first term $\left| \nabla (\frac{\rho}{A}) \right|_\infty \leq \left(C_1 |\nabla \rho|_\infty + |\rho|_\infty |\nabla A|_\infty C_1^2 \right)$. Therefore, the above inequality can be bounded by:

$$\begin{aligned}
\sum_{|\alpha| \leq m} I_\alpha &\lesssim \| J_\epsilon \nabla \rho \|_m \left(C_1 |\nabla \rho|_\infty + |\rho|_\infty |\nabla A|_\infty C_1^2 \right) \| A \|_m \\
&\quad + \| J_\epsilon \nabla \rho \|_m \left(|J_\epsilon \nabla A|_\infty \left\| D^m \frac{\rho}{A} \right\|_0 + \left| \frac{\rho}{A} \right|_\infty \| J^\epsilon \nabla A \|_m \right) .
\end{aligned} \tag{A.2}$$

The term $\left\|D^m \frac{\rho}{A}\right\|_0$ can be bounded by simpler terms using part *(1)* of *Lemma 2.2*. In particular,

$$\left\|D^m \frac{\rho}{A}\right\|_0 \leq c\left(|\rho|_\infty \left\|D^m \frac{1}{A}\right\|_0 + C_1 \|D^m \rho\|_0\right). \tag{A.3}$$

Here we make use of *Lemma Appendix A.1* by substituting (A.1) into (A.3). From (A.2) after applying a Cauchy inequality of the form $2ab \leq \delta a^2 + \frac{1}{\delta}b^2$ we get the desired result. □

A.3. *Computations for L^2-Cauchy Sequence*

$$\begin{aligned}
R_1 &\lesssim \left(\left|\frac{1}{A^\epsilon}\right|_\infty \|D(\nabla\rho \cdot J_\epsilon \nabla A^\epsilon)\|_0 + |\nabla\rho^\epsilon|_\infty |\nabla J_\epsilon A^\epsilon|_\infty \left\|D\left(\frac{1}{A^\epsilon}\right)\right\|_0\right) \\
&\lesssim \left|\frac{1}{A^\epsilon}\right|_\infty \left\{|J_\epsilon \nabla A^\epsilon|_\infty \|\Delta\rho^\epsilon\|_0 + \left\|D^2 J_\epsilon A^\epsilon\right\|_0 |\nabla\rho^\epsilon|_\infty\right\} + \left|\frac{1}{A^\epsilon}\right|_\infty^2 |\nabla\rho^\epsilon|_\infty |\nabla J_\epsilon A^\epsilon|_\infty \|\nabla A^\epsilon\|_0 \\
&\leq c\left(\left|\frac{1}{A^\epsilon}\right|_\infty + \left|\frac{1}{A^\epsilon}\right|_\infty^2\right) \|A^\epsilon\|_3 \left(\|\rho^\epsilon\|_3 + \|A^\epsilon\|_3 \|\rho^\epsilon\|_3\right)
\end{aligned}$$

similarly,

$$\begin{aligned}
R_2 &\leq c\left(\left|\frac{1}{A^\epsilon}\right|_\infty \|D(\rho^\epsilon \Delta J_\epsilon A^\epsilon)\|_0 + |\rho^\epsilon \Delta J_\epsilon A^\epsilon|_\infty \left\|D\left(\frac{1}{A^\epsilon}\right)\right\|_0\right) \\
&\leq c\left|\frac{1}{A^\epsilon}\right|_\infty \left(\|\rho^\epsilon\|_3^2 + \|A^\epsilon\|_4^2\right)
\end{aligned}$$

Computations for *Lemma 4.5*

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int v^2 dx &= \int v\left[\Delta v - 2\nabla \cdot \left(\frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2\right) - A_1\rho_1 + A_2\rho_2\right] dx \\
&\overset{\text{C.I}}{\leq} \left\|\frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2\right\|_0^2 - \int A_2 v^2 dx - \int \rho_1 uv dx \\
&\leq \left\|\frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2\right\|_0^2 + \frac{1}{2}|\rho_1|_\infty \|u\|_0^2 + \frac{1}{2}|\rho_1|_\infty \|v\|_0^2
\end{aligned}$$

Unfortunately, the advection term leaves a term which still has to be dealt with:

$$\begin{aligned}
\left\|\frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2\right\|_0^2 &\leq \left|\frac{\rho_1}{A_1}\right|_\infty^2 \|\nabla u\|_0^2 + |\nabla A_2|_\infty^2 \left\|\frac{A_2\rho_1 - A_1\rho_2}{A_1 A_2}\right\|_0^2 \\
&\leq \left|\frac{\rho_1}{A_1}\right|_\infty^2 \|\nabla u\|_0^2 + |\nabla A_2|_\infty^2 \left|\frac{1}{A_1 A_2}\right|_\infty^2 \left(|\rho_1|_\infty^2 \|u\|_0^2 + |A_1|_\infty^2 \|v\|_0^2\right).
\end{aligned}$$

Making use of the fact that $|1/A|_\infty \leq C_1$ gives the final result.

A.4. *Extended Sobolev Inequalities*

For the proof of the following theorem see [17].

Theorem Appendix A.1 (Extended Sobolev Inequalities in Bounded Domains). *Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let u be any function in $W^{m,r}(\Omega) \cap L^p(\Omega)$, $1 \leq r, q \leq \infty$. For any integer j, $0 \leq j \leq m$, and for any number a in the interval $j/m \leq a \leq 1$, set*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

If $m - j - n/r$ is a nonnegative integer, then

$$\left\|D^j u\right\|_{L^p} \leq C \left\|u\right\|_{W^{m,r}}^{a} \left\|u\right\|_{L^q}^{(1-a)}. \tag{A.4}$$

If $m - j - n/r$ is a nonnegative integer, then (A.4) *holds for $a = j/m$. The constant C depends only on Ω, r, q, m, j a.*

Deriving Inequality (5.8)
Applying (A.4) for $p = 2$ gives:

$$\begin{aligned}\left\|u\right\|_{L^2}^2 &= C \left\|u\right\|_{W^{1,2}} \left\|u\right\|_{L^1} \\ &\leq \epsilon \left(\left\|u\right\|_{L^2}^2 + \left\|\nabla u\right\|_{L^2}^2\right) + \frac{C}{\epsilon} \left\|u\right\|_{L^1}^2\end{aligned}$$

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