Nonlinear regularizations of TV based PDEs for image processing

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ABSTRACT. We consider both second order and fourth order TV-based PDEs for image processing in one space dimension. A general class of nonlinear regularizations of the TV functional result in well-posed uniformly parabolic equations in two dimensions. However the fourth order analogue (Osher et. al. *Multiscale Methods and Simulation* 1(3) 2003) based on a total variation minimization in an H^{-1} norm, has very different properties. In particular, nonlinear regularizations should have special structure in order to guarantee that the regularized PDE does not produce finite time singularities.

1991 *Mathematics Subject Classification.* Primary 35G25; Secondary 68U10, 94A18, 35Q80.

AB and JG are supported by ONR grants N000140410078 and N000140410054, NSF grants ACI-0321917 and DMS-0244498, and ARO grant DAAD19-02-1-0055. This paper is based upon work supported by the National Science Foundation and the intelligence community through the joint “Approaches to Combat Terrorism” program (NSF Grant DMS-0345602). We thank Tony Chan for useful comments.

Nonlinear PDEs are now commonly used in image processing for issues ranging from edge detection, denoising, and image inpainting, to texture decomposition. Second order PDEs for image denoising and boundary or edge sharpening date back to the seminal works of Rudin-Osher-Fatemi [**14**], and Perona-Malik [**13**]. All of these methods have some common features; they are based on a nonlinear version of the heat equation

$$u_t = \nabla \cdot ((g(|\nabla u|)\nabla u) \tag{0.1}$$

in which the 'thresholding function' g is small in regions of sharp gradients. A number of mathematical issues arise with these equations and their use. For example, Perona-Malik, suggest using a smooth g that decays fast enough for large ∇u so that significant diffusion only takes place in regions of small change in the image, i.e. away from edge boundaries. A typical choice might be

$$g(s) = k^2/(k^2 + s^2). \tag{0.2}$$

However, this and similar choices result in a PDE that is linearly ill-posed in regions of high gradients and the ensuing dynamics results in a characteristic "staircase" instability.

A particular class of denoising algorithms are the TV (total variation) methods introduced by Rudin, Osher and Fatemi [**14**]. The technique minimizes the total variation norm of the image. The TV functional is defined as

$$TV(u) = \int_\Omega |\nabla u|. \tag{0.3}$$

The TV functional does not penalize discontinuities in u and thus allows one to recover the edges of the original image. The restoration problem can be written as

$$\min_u \int_\Omega (|\nabla u| + \frac{\lambda}{2}(u-f)^2). \tag{0.4}$$

To solve the minimization problem, one typically writes down the Euler-Lagrange equation and performs a gradient descent. For the above problem, this means solving the nonlinear PDE

$$u_t = \nabla \cdot (\frac{\nabla u}{|\nabla u|}) + \lambda(f-u). \tag{0.5}$$

In the past few years, a number of authors have proposed analogous fourth order PDEs for the same functions (i.e. edge detection, image denoising, etc.) with the hope that these methods might perform better than their second order analogues. Indeed there are good reasons to consider fourth order diffusions. First, fourth order linear diffusion damps oscillations at high frequencies (i.e. noise) much faster than second order diffusion. Second, there is the possibility of having schemes that include effects of curvature (i.e. the second derivatives of the image) in the dynamics, which opens up a richer set of functional behaviors. On the other hand, the theory of fourth order nonlinear PDEs is much less well-developed than their second order analogues. Also, such equations often do not possess a maximum principle or comparison principle and thus allow for the possibility of artificial singularities or undesirable behavior in their implementation.

This paper contrasts the well-known second order TV methods with the fourth order PDE derived by Osher, Solé and Vese [**12**] for texture-noise decomposition using TV minimization in the H^{-1} norm:

$$u_t = -\Delta[\nabla \cdot (\frac{\nabla u}{|\nabla u|})] - \lambda(u-f). \tag{0.6}$$

1. Texture-Cartoon decomposition and fourth order PDEs

An important problem in image processing is to separate textures from larger scale features in images. In [**11**] Yves Meyer suggested replacing the ROF model by

$$u = \operatorname{argmin} \left(\int |\nabla u| + \lambda \|f-u\|_* \right). \tag{1.1}$$

Here the * norm corresponds to the space G, the dual of BV. This should enable us to get a model which does not smear textures. Instead, at the scale λ, u is a cartoon version of f and $f-u=v$ consists of texture plus noise.

This is a beautiful idea, but apparently difficult to implement because the $*$ norm is fairly complicated. Picking $g=(g_1,g_2)$ so that

$$v \quad = \quad (g_1)_x + (g_2)_y = \nabla \cdot g \qquad \text{(in two dimensions)},$$

we define

$$\|v\|_* \quad := \quad \inf_g \sup_{(x,y)} \sqrt{g_1^2+g_2^2}.$$

Thus, the variational problem (1.1) does not have a simple Euler-Lagrange PDE which could be used to find the minimum.

In [**16**] an approximation to Meyer's model was obtained

$$\begin{aligned}(u,g_1,g_2) = \quad & \operatorname{argmin} \int |\nabla u| + \mu \int |f-(u+\partial_x g_1 + \partial_y g_2)|^2 \\ & + \lambda \left(\int (\sqrt{g_1^2+g_2^2})^p \right)^{\frac{1}{p}}.\end{aligned}$$

As $\mu, p \to \infty$, this model approaches Meyer's. This variational problem gives very good texture/cartoon separation. It is a model of the form $f=u+v+w$, u is cartoon, $v=\nabla\cdot g$ is texture and w becomes small as μ increases.

An $f=u+v$ approximation to Meyer's model was obtained in [**12**] as follows: One writes

$$f = u+v = u+\nabla\cdot g$$

then uses the Hodge decomposition theorem to decompose

$$g = \nabla p + w$$

where

$$\nabla \cdot w = 0.$$

This means that $g=\nabla\Delta^{-1}(f-u)$ (see [**12**] for more details). In order to obtain a local Euler-Lagrange equation to minimize the model, the following functional was chosen

$$u = \operatorname{argmin} \int |\nabla u| + \frac{\lambda}{2} \int |\nabla \Delta^{-1}(f-u)|^2. \tag{1.2}$$

The resulting Euler-Lagrange equation is

$$-\nabla \cdot \frac{\nabla u}{|\nabla u|} + \lambda \Delta^{-1}(f-u) = 0.$$

It was shown in [**12**] that the resulting minimizer of (1.2) is the solution of equation (0.6) as $t\to\infty$. This model appears to give the best denoising results of the three models, while preserving edges. It also apparently separates texture from cartoon, but not as well as the model in [**16**] does.

In this paper we discuss regularizations of the singular nonlinearity in (0.6) which is an important problem for the numerical implementation of these methods. We show that special care must be taken in the choice of regularizing function. This same issue does not arise for second order equations due to a maximum principle. In the case of fourth order nonlinear diffusion equations it is the structure of the nonlinearity that determines whether a weak maximum principle can be derived; hence effecting the choice of regularization.

2. Well-posedness of second order TV regularizations

It is common practice in numerical implementations of the second order equation (0.5) to regularize the jump singularity where ∇u vanishes. For example, Vogel and Oman [**17**], Marquina and Osher [**10**], and Chan et al [**5, 6**] all consider the following regularization of the PDE:

$$u_t = \nabla \cdot \left(\frac{\nabla u}{(\delta^2 + |\nabla u|^2)^{1/2}}\right) + \lambda(f-u) \tag{2.1}$$

where δ is a smoothing parameter. This has the advantage of being a PDE with classical smooth solutions. This particular choice of regularization also arises in data analysis; it is the "Huber function," or "Huber norm," which interpolates between smooth (least-squares) measures and robust L^1 error measures [**9**].

In fact it does not matter so much how one chooses the regularization provided that it is monotone and smooth. Below we review the derivation of these equations from a variational formulation.

2.1. Variational form for the regularization.

Consider the variational problem

$$u^* = \text{argmin} \quad \mathrm{F(u)} \equiv \int |\nabla \mathrm{u}| \mathrm{dx} + \frac{\lambda}{2} ||\mathrm{f} - \mathrm{u}||_2^2, \tag{2.2}$$

in which we minimize total variation without deviating too wildly from the measured data f. Differentiating the right hand side, we obtain the corresponding Euler-Lagrange equation. In one dimension we get

$$DF(u) = -\left(\frac{u_x}{|u_x|}\right)_x - \lambda(f-u) = 0. \tag{2.3}$$

In two or more dimensions we have

$$DF(u) = -\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) - \lambda(f-u) = 0 \tag{2.4}$$

with the corresponding gradient descent PDE

$$u_t = -DF(u) = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(f-u). \tag{2.5}$$

Notice that we now have a highly non-regular equation: in one dimension (2.3) is nothing more than $DF(u) = -\phi(u_x) - \lambda(f-u)$ where $\phi()$ is the Dirac delta "function". To regain some regularity we modify the variational problem as follows:

$$u^* = \text{argmin} F(u) \equiv \int \sqrt{\nabla u \cdot \nabla u + \delta^2} dx + \frac{\lambda}{2} ||f-u||_2^2 \tag{2.6}$$

where δ is a small parameter (and $\delta = 0$ corresponds to the unregularized version in equation 2.2). Differentiating, we obtain

$$\begin{aligned} u_t = & \quad \nabla \cdot \left(\frac{\nabla u}{\sqrt{\nabla u \cdot \nabla u + \delta^2}}\right) + \lambda(f-u) && (2.7) \\ = & \quad \nabla \cdot H_\delta(\nabla u) + \lambda(f-u) && (2.8) \end{aligned}$$

where $H_\delta(v) = \frac{v}{\sqrt{v \cdot v + \delta^2}}$. This is equation (2.1) in arbitrary dimension. The choice of regularizing functional is somewhat arbitrary. As we discuss in the next section, almost any 'nice' choice of regularization leads to a PDE that satisfies a maximum principle and thus has globally smooth solutions. This trait is special for second order PDEs. When we consider the fourth order analogues of these problems, we will see that the choice of regularization affects the dynamics of the gradient descent.

Finally note that for the above to be useful, it should apply (since image gray levels are often discontinuous) to discontinuous u for which u_x and ∇u are Radon measures. A detailed discussion of the space of functions of bounded variation, BV, can be found in Evans and Gariepy [**7**]. For the related Mumford-Shah functional see Ambrosio, Fusco and Pallara [**1**]. We also mention the notes by Chambolle [**4**] which are available online.

2.2. Maximum principle. Following the classical theory of nonlinear second order parabolic PDEs, we see that equation (2.7) admits global smooth solutions provided that the gradient of the solution, ∇u, can be shown to stay bounded for all time. This fact is a result of the maximum principle for second order equations. The following lemma can be proved for (2.7) in one space dimension with a variety of standard boundary conditions used in imaging, including Neumann and periodic.

LEMMA 1. *(A priori bound for the slope) Consider a smooth solution of (2.7) in one space dimension, then the gradient is a priori bounded in time by max* $(|f_x|, |u_{0x}|)$.

In practice of course the observed data f may be quite jumpy, possibly possessing a singular gradient. The point of this exercise is to show that jump discontinuities can not possibly form spontaneously due to the minimizing flow, that they have to come from the matching to the observed data.

Proof. Let $w = u_x$ denote the slope. Then the equation for w is

$$w_t = H''_\delta(w) w_x^2 + H'_\delta(w) w_{xx} + \lambda(f_x - w). \tag{2.9}$$

If H_δ is smooth and monotone in w then the equation satisfies a maximum principle. Standard arguments then complete the proof.

3. The 1D H^{-1} equation with smoothing, some examples

Behavior of the fourth order PDE in one dimension is very relevant for two dimensional images. This is because a lot of the structure involves edges and information separated by edges, which are basically one-dimensional objects. Moreover, we can obtain a lot of insight about the dynamics in 1D by combining ideas from numerics, asymptotics, and rigorous analysis that may not be so tractable in 2D (given things like energy estimates and the Sobolev lemma).

In one space dimension we have the PDE

$$u_t = -[\frac{u_x}{|u_x|}]_{xxx} - \lambda(u - f). \tag{3.1}$$

As for the classical ROF model, we can solve this numerically by using a regularization of the signum function. Below we discuss some special cases and the consequence of using different choices for the smoothing function.

3.1. Example: Arctan regularization. Consider replacing $\frac{u_x}{|u_x|}$ in (3.1) with $\frac{2}{\pi}\arctan(u_x/\delta)$ where δ is the smoothing parameter. Then the PDE becomes

$$u_t = -\frac{2}{\pi}[\arctan(u_x/\delta)]_{xxx} - \lambda(u-f). \tag{3.2}$$

We now introduce a new variable $w = \arctan(u_x/\delta)$ and rewrite the PDE as

$$\delta(\tan w)_t = -\frac{2}{\pi} w_{xxxx} - \lambda(\delta \tan w - f_x) \quad . \tag{3.3}$$

which we can rewrite (to remove the $\cos w$ singularity) as

$$\delta(w)_t = -\frac{2}{\pi}\cos^2 w w_{xxxx} - \lambda(\delta \sin w \cos w - \cos^2 w f_x) \quad . \tag{3.4}$$

This is a variant of a PDE that arises in a modified lubrication equation (MLE) for thin films [**2**],

$$u_t = -u^n u_{xxxx} \tag{3.5}$$

and as the equation for the smoothness estimator in Low Curvature Image Simplifiers [**3**]. Results for (3.5) in one dimension are that smooth positive data gives a smooth positive solution for all time if $n \geq 5/3$. Numerical and asymptotic results suggest that $n = 3/2$ is the critical exponent, above which finite time singularities do not occur and below which they do occur. These results also imply similar results for fourth order degenerate diffusion equations with the same structure, i.e. a degenerate nonlinearity in which u^n is replaced by $f(u)$, here it is $\cos^2 u$. Thus the degeneracy here corresponds to the case $n = 2$ in (3.5) for which we have a theorem that singularities can not occur.

We have the following theorem, which can be proved following the arguments in [**3**], which is essentially the same equation as (3.4) (note the only new terms are the ones from the fidelity, which are mild):

THEOREM. *Equation (3.4) with smooth initial data and forcing function f has a unique smooth solution for all time.*

Remark: Transforming back to the original PDE we see that this implies well-posedness with a priori bounds on u_x, i.e. the slope. This is actually relevant for solutions with rough data, i.e. the bound on the slope is a bound related to the initial data and the forcing. The resulting dynamics insure that the solution of the PDE can not become singular from the dynamics of the gradient descent, as in the general case for the second order equations discussed in the previous section.

3.2. Example: square root smoothing. We show that a different smoothing can have a different result with respect to well-posedness. Consider instead of the arctan above the substitution $\frac{u_x}{(u_x^2+\delta^2)^{1/2}}$, following the conventional approach in image processing. Following the same argument as in the previous example, we have an equation for $w = \frac{u_x}{(u_x^2+\delta^2)^{1/2}}$:

$$\delta(\frac{w\delta}{(1-w^2)^{1/2}})_t = -w_{xxxx} - \lambda(\frac{w\delta}{(1-w^2)^{1/2}} - f_x) \quad . \tag{3.6}$$

Note that this produces a degeneracy in the w equation at ± 1 that has a $3/2$ power, i.e. it is analogous to (3.5) with $n = 3/2$. For this problem we do not have a theorem that guarantees well-posedness, however careful empirical studies of forced singularities in equation (3.5) [**2**] show that they happen in infinite time, not finite time, which suggests that this regularization could have practical use.

From these last two examples we see that the behavior of the regularized PDE, with respect to the possibility of finite time singularities (in u_x), is related to the rate at which the regularizing function approaches a constant in the far field. I.e. we substitute $f(u_x/\delta)$ for $u_x/|u_x|$ and the issue is how fast does f approach ± 1 as $u_x \to \infty$. In the arctan case we have a quadratic nonlinearity in the w equation (from taking a derivative of the inverse). In the square root case the decay is like $1/|u_x|^{1/2}$ which gives us a $3/2$ power degeneracy in the w equation.

3.3. Example: tanh smoothing. Consider instead of the arctan above the substitution $\tanh(u_x/\delta) = w$. Following the same argument as in the previous example, we have an equation

$$\delta(\tanh^{-1} w)_t = -w_{xxxx} - \lambda(\delta \tanh^{-1} w - f_x) \quad . \tag{3.7}$$

which gives

$$\delta(w)_t = -(1-w^2)w_{xxxx} - \lambda(\delta(1-w^2)\tanh^{-1} w - f_x) \quad . \tag{3.8}$$

Note that here the degeneracy at $w = \pm 1$ is linear, which suggests that this regularization has finite time singularities, perhaps making it a bad choice for a numerical method.

4. General smoothing functions and well-posedness in 1D

In this section we consider a general class of smoothing functions for the fourth order equation and show that for a subset of such functions we can guarantee well-posedness of the resulting regularized PDE. We conjecture that there are smoothing functions within the general class that do not produce globally smooth solutions. This is in sharp contrast to the second order case for which the maximum principle guarantees well-posedness of all smooth monotone regularizations.

Consider the regularized equation

$$u_t = -\left(H(u_x)\right)_{xxx}, \tag{4.1}$$

which we interpret as a nonlinear regularization of (3.1). We set $\lambda = 0$ to simplify the analysis and to focus on the nonlinearity in the gradient descent. The results here extend directly to the case with nonzero λ.

Define

$$\gamma := \int_0^\infty \frac{1}{(1+s^2)^\alpha} \tag{4.2}$$

and

$$H(y) := \frac{1}{\gamma}\int_0^y \frac{1}{(1+s^2)^\alpha}, \tag{4.3}$$

we see that for $\alpha > \frac{1}{2}$, H is a regularization of the Heaviside function. $\alpha = \frac{3}{2}$ is the typical choice of regularization for TV. $\alpha = 1$ gives an arctan regularization. In this section we prove the following theorem

THEOREM. *Consider smooth initial data u_0 and (4.1) on a periodic interval. Then for all $1/2 < \alpha \leq 5/4$, there exists a unique smooth solution of (4.1) for all time $t > 0$. Moreover, the solution is bounded away from the singular values ± 1 by a fixed constant independent of t.*

In this paper we prove the key part of the theorem, namely the a priori bound. Local existence and continuation in time can be proved following the arguments in [**3**] for a related image processing equation. We are unable to prove such a bound for all $\alpha > 1/2$ due to the nonlinear structure of the equation; values of alpha described above correspond precisely to $n > 5/3$ in the analogous result for (3.5). Moreover we conjecture that for sufficiently large α the regularized equation (4.1) admits solutions with finite time singularities even if the forcing function f is smooth. Note that here we have set $\delta = 1$ for simplicity of notation, although the analysis holds with other values of δ.

We will be interested in the behavior of $H^{-1}(w)$ for w near ± 1, which corresponds to large y in $H(y)$. Using $A \sim B$ to mean that there exists constants C_1 and C_2 such that $A \leq C_1 B$ and $B \leq C_2 A$, we have

$$1 - H(y) \sim \frac{1}{y^{2\alpha-1}} \tag{4.4}$$

for positive $\infty > y >> 1$ and

$$-1 - H(y) \sim -\frac{1}{y^{2\alpha-1}} \tag{4.5}$$

for $-\infty < y << -1$. It follows that

$$\left|H^{-1}(w)\right| \sim \frac{1}{|1-|w||^{\frac{1}{2\alpha-1}}}$$

for w near ± 1. For $\alpha = 1$,

$$H(y) = \frac{2}{\pi}\arctan y,$$

$$H^{-1}(w) = \tan\left(\frac{\pi}{2} w\right),$$

and

$$\tan(\pi w/2) \sim \frac{1}{|1-|w||}$$

for w near ± 1.

4.1. The change of variables. Consider the equation $u_t = -\left(H(u_x)\right)_{xxx}$. Defining $w = H(u_x)$ and differentiating once with respect to x, the equation becomes

$$\left(H^{-1}(w)\right)_t = -w_{xxxx}. \tag{4.6}$$

Using the property of general invertible functions f,

$$\left(f^{-1}(u)\right)' = \frac{1}{f'\left(f^{-1}(u)\right)},$$

we rewrite (4.6) as

$$w_t = -H'\left(H^{-1}(w)\right) w_{xxxx}. \tag{4.7}$$

For $\alpha = 1$,

$$H'\left(H^{-1}(w)\right) = \frac{1}{1 + \tan^2(\pi w/2)} = \cos^2(\pi w/2),$$

so the corresponding equation is

$$w_t = -\cos^2(\pi w/2)\, w_{xxxx}.$$

4.2. Relationship with the modified lubrication equation (3.5)**.** Since

$$H'(H^{-1}(w)) = \frac{1}{\left(1+(H^{-1}(w))^2\right)^{\alpha}},$$

for w near 1, (4.4) gives us

$$H'\left(H^{-1}(w)\right) \sim (1-w)^{\frac{2\alpha}{2\alpha-1}}$$

and a similar result for w near -1. Thus near any singularity, (4.6) behaves like (3.5) for

$$n = \frac{2\alpha}{2\alpha-1}. \tag{4.8}$$

Since Bertozzi proved that solutions to (3.5) can be continued for all time when $n \geq \frac{5}{3}$, we expect a similar result for (4.6), when $\frac{1}{2} < \alpha \leq \frac{5}{4}$ (Note that in (4.8), $n \to \infty$ as α decreases to $\frac{1}{2}$). We remind the reader that $\alpha = \frac{1}{2}$ does not give a regularized step function H, since the integral in (4.2) is infinite in that case.

4.3. A priori bounds. In all that follows, we assume periodic boundary conditions on $[0,1]$. Taking the L^2-inner product of (4.7) with w_{xxxx} gives

$$\frac{d}{dt} \parallel w_{xx} \parallel_0^2 = -\int H'\left(H^{-1}(w)\right)(w_{xxxx})^2 \leq 0. \tag{4.9}$$

Integrating over time then gives

$$\int_0^T \int H'\left(H^{-1}(w)\right)(w_{xxxx})^2 \leq \parallel (w_0)_{xx} \parallel_0^2 . \tag{4.10}$$

Since w is assumed to be in $[-1,1]$, we have $w \in H^2$. The Sobolev Embedding Theorem gives $w \in C^{1,\frac{1}{2}}$.

4.4. Entropy. We now find an integral that remains bounded for this equation, but would necessarily blow up if $w \to \pm 1$. Consider

$$\Psi = \int wH^{-1}(w) \geq 0.$$

$$\begin{aligned}
\frac{d}{dt}\int wH^{-1}(w) &= \int w_t H^{-1}(w) + \int w\left(H^{-1}(w)\right)_t \\
&= -\int H^{-1}(w)H'\left(H^{-1}(w)\right)w_{xxxx} - \int ww_{xxxx} \\
&\leq C\left|H^{-1}(w)H'\left(H^{-1}(w)\right)\right|_{L^\infty}^{\frac{1}{2}} \left(\int H^{-1}(w)\right)^{\frac{1}{2}} \left(\int H'(H^{-1}(w))(w_{xxxx})^2\right)^{\frac{1}{2}}
\end{aligned}$$

We note that

$$yH'(y) = \frac{y}{(1+y^2)^{\alpha}},$$

so we have a bound

$$\left| H^{-1}(w)\left(H'\left(H^{-1}(w)\right)\right)^{\frac{1}{2}} \right|_{L^\infty} \leq 1 \tag{4.11}$$

as long as $\alpha \geq \frac{1}{2}$. Also since $H^{-1}(w)$ is bounded for small w,

$$\int H^{-1}(w) \leq C\left(1 + \int wH^{-1}(w)\right),$$

and we therefore have

$$\frac{d}{dt}\Psi(t) \le C_1\Psi(t) + C_2\left(1 + \int H'(H^{-1}(w))\,(w_{xxxx})^2\right). \tag{4.12}$$

Grönwall's inequality gives

$$\Psi(T) = \int wH^{-1} \le \left(\Psi(0) + C_2T + \int_0^T \int H'(H^{-1}(w))\,(w_{xxxx})^2\right) e^{C_1T} \tag{4.13}$$

and the a priori estimate (4.10) implies that $\Psi(T)$ is bounded.

4.5. Uniform bound on w**.** A priori estimates give $w \in C^{\frac{3}{2}}$, so given a maximum value $\eta(t)$ of w at $x_0(t)$ for a particular time t, there is some constant K satisfying

$$w(x,t) \ge \eta(t) - K\,|x - x_0(t)|^{\frac{3}{2}}. \tag{4.14}$$

Assuming existence of w on $[0,T)$, pick $t \in [0,T)$. Then

$$\begin{aligned}
C &\ge \int wH^{-1}(w)\,dx \\
&\ge C_1 \int \frac{1}{(1-w)^{\frac{1}{2\alpha-1}}}dx \\
&\ge C_1 \int \frac{1}{\left(1-\eta+K|x-x_0|^{\frac{3}{2}}\right)^{\frac{1}{2\alpha-1}}}dx \\
&= C_1\,(1-\eta)^{\frac{-1}{2\alpha-1}} \int \frac{1}{\left(1+\frac{K}{1-\eta}\,|x-x_0|^{\frac{3}{2}}\right)^{\frac{1}{2\alpha-1}}}dx \\
&\ge C_2\,(1-\eta)^{\frac{2}{3}-\frac{1}{2\alpha-1}} \int_0^{\frac{K}{2}(1-\eta)^{-\frac{2}{3}}} \frac{1}{\left(1+y^{\frac{3}{2}}\right)^{\frac{1}{2\alpha-1}}}dy
\end{aligned}$$

So

$$(1-\eta)^{\frac{2}{3}-\frac{1}{2\alpha-1}} \le C_3,$$

which gives a bound M with $\eta \le M < 1$ as long as

$$\frac{2}{3} - \frac{1}{2\alpha-1} < 0,$$

or

$$\alpha < \frac{5}{4}. \tag{4.15}$$

Similar steps bound η away from 1 for the case $\alpha = \frac{5}{4}$. We can therefore follow the arguments in [**3**] to prove existence of solutions to (4.7) globally in time for $\frac{1}{2} < \alpha \le \frac{5}{4}$. Essentially the same arguments bound η away from -1 for the same range of α.

5. Conclusions

Total variation based algorithms have led to an interesting class of nonlinear PDEs in image processing. The second order equations have been in use for over ten years with much success. Their numerical implementation often requires a regularization of the nonlinearity in the TV functional. Although there is a common way to do this, the choice of regularization is not so important due to the maximum principle for second order parabolic equations. That is, essentially all reasonable choices of regularizations lead to PDEs that are globally well-posed for smooth initial data.

More recently there has been an interest in higher order equations in image processing. Here we consider a method introduced in [**12**] motivated by the work of Meyer [**11**]. The equation (0.6) is a fourth order analogue of the now classical ROF method ([**14**]). In this paper we focus on the one dimensional version of (0.6), showing that one can not in general regularize the TV functional in (0.6) and be guaranteed of a well-posed problem. The issue is that the maximum principle does not hold for fourth order equations. However, using ideas from the thin films literature, in particular [**2**], we show that there is a subclass of nonlinear regularizations for which the PDE is well-posed. The point is that the structure of the differential operator does not lead to a maximum principle as in the second order case, however the structure of the nonlinearity results in a weak maximum principle (bounding the solution away from the singular values) for carefully chosen nonlinearities.

Although our results focus on the problem in one dimension, we believe that this idea has relevance to higher dimensions. Recently, two of the authors [**3**] have used this idea to prove global well-posedness of a related image processing problem known as Low Curvature Image Simplifiers, originally introduced by Tumblin and Turk [**15**]. While well-posedness was proved in one space dimension, the idea behind the design of the scheme works for two dimensions as well. Other related fourth order imaging equations are believed to produce singularities [**8**] and we believe this issue is critical in understanding how to design and implement higher order methods in imaging.

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