Geometric interpretation of Determinants

- We can view determinants as how much a linear transformation scales signed areas (volumes).

Example: Consider $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = A$. What happens to the unit square after applying $A$? is $S_2 = [0, 1] \times [0, 1]$.

So it maps to a square of area $4$. det$(A) = 9$.

Note

- This works for all shapes, not just squares.
- This is a "signed area". If the transformation swaps the orientation of the basis vectors, the area is negative.

**Question 1.** For the following matrices determine it's determinant geometrically. That is, draw the image of the square $[0, 1] \times [0, 1]$ under the transformation and calculate it's signed area.

(a) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  
(b) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$  
(c) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
(a) Image: This has area 2
\[ \det(A) = 1. \]

(b) Image: Thus has area 2
Hence \[ \det(A) = 2. \]

(c) Image: The orientation has been flipped, so the signed area is -1.
\[ \det(A) = -1. \]

How do we calculate determinants algebraically? There are lots of ways such as:
There are 10 ways such as

- Laplace expansion
- Permutation definition

but the easiest is Sarrus' rule.

Example:

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix}
\]

\text{det}(A) = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9

= -3.

Question 2. Use Sarrus' rule to calculate the determinant of the matrix

\[
\begin{pmatrix}
4 & 2 & 0 \\
4 & 6 & 0 \\
5 & 2 & 3 \\
\end{pmatrix}
\]

\[
\text{det}(A) = 4 \cdot 6 \cdot 3 + 2 \cdot 0 \cdot 5 + 0 \cdot 4 \cdot 2
- 5 \cdot 6 \cdot 0 - 2 \cdot 0 \cdot 3 - 3 \cdot 4 \cdot 2
= 72 - 24
= 48.
\]
Question 3. (Discussion) Determinants are additive in their columns. For instance, we have that
\[
\det \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix} = \det \begin{pmatrix} 3 + 2 & 0 \\ 3 - 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix} + \det \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}.
\]
Can you explain why this is true geometrically?

\[
\det A = \text{area of the parallelogram determined by its columns}. \quad \text{If we take two parallelograms with one side parallel to each other and stick them together along this side, this has the same area as the parallelogram with one side the same and the other side the vector addition of the other sides of the original.}
\]
Question 4. (Discussion) Which of the following are true, which are false?

(a) \( \det(A + B) = \det(A) + \det(B) \) for all 5 \( \times \) 5 matrices A and B.
(b) \( \det(A) = \det(-A) \) for all 6 \( \times \) 6 matrices A.
(c) If A is an invertible matrix, then \( \det(A^T) = \det(A^{-1}) \).
(d) The determinant of an orthogonal matrix is 1.

(e) There exists an invertible matrix of the form

\[
\begin{pmatrix}
a & e & f & j \\
b & 0 & g & 0 \\
c & 0 & h & 0 \\
d & 0 & i & 0 \\
\end{pmatrix}
\]

(a) False. While the columns and rows are additive, the entire matrices aren't.

\[ \det \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = 2 \text{ while } \det \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0. \]

(b) True. We can take scalars out of rows/columns.

So we get \( \det(-A) = (-1)^6 \det(A) = \det(A) \).

(c) False. We have \( \det(A) = \det(A^T) \) and

\[ \det(A^{-1}) = \frac{1}{\det(A)} \text{ in general.} \]

(d) True. Geometrically, orthogonal transformations don't change area.

(e) \( \det \left( \begin{pmatrix} a & e & f & j \\ b & 0 & g & 0 \\ c & 0 & h & 0 \\ d & 0 & i & 0 \end{pmatrix} \right) = -e \left| \begin{pmatrix} b & g & 0 \\ c & h & 0 \\ d & i & 0 \end{pmatrix} \right| \text{ by Laplace expansion along second column} \]

\[ = 0. \]
since $\det(A) = 0$, can't be invertible.