Orthogonal transformations $T$ are "rigid" transformations. By this I mean they are really only rotations and reflections. They do not stretch or shrink vectors when applied to them. We have a few characterizations of orthogonal transformations.

**Theorem:** Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a transformation with corresponding matrix $A$, i.e. $T(x) = Ax$. Then the following conditions are equivalent:

1. The columns of $A$ form an orthonormal basis for $\mathbb{R}^n$.

2. The transformation preserves length. That is $||T(x)|| = ||x||$.

3. The transformation preserves dot products. That is $T(x) \cdot T(y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$.


5. $A^T = A^{-1}$.

Any transformation that satisfies any of these conditions is called an orthogonal transformation.
Question 1. (Discussion)

Given orthogonal matrices $A$ and $B$. Which of the following are always orthogonal? For those not orthogonal, can you give a counterexample?

(a) $AB$
(b) $A + B$
(c) $B^{-1}$

(a) $AB$ geometrically is composing two rigid transformations. Hence this is a rigid transformation and so orthogonal.

A more rigorous proof can use the length characterization
\[ \| ABx \| = \| A(Bx) \| = \| Bx \| = \| x \| . \]

(b) This is not in general orthogonal. Take $A = B = I$

Then $A + B = 2I$ which is not orthogonal.

\[ \| 2I x \| = \| 2 x \| = 2 \| x \| \neq \| x \|. \]

(c) $B^{-1}$ is orthogonal. Undoes a rigid transformation.

Alternatively, since $B$ is orthogonal it is invertible.

And so has full rank. For all $x \in \mathbb{R}^n$ there exists $\tilde{y} \in \mathbb{R}^n$ such that $x = A \tilde{y} \Rightarrow A^{-1} x = \tilde{y}$.

Hence $\| A^{-1} x \| = \| \tilde{y} \| = \| A \tilde{y} \| = \| A A^{-1} x \| = \| x \|$. This holds for all $x$ and so $A^{-1}$ orthogonal.

Question 2. (Discussion)

Suppose we have a basis $v_1, \ldots, v_r$ of a subspace $V$. After Gram-Schmidt, we find an orthonormal basis $u_1, \ldots, u_r$. From this we have the QR-factorization $M = QR$ where $M$ is the matrix with columns the $v_i$ and $Q$ with columns $u_i$. How do we recover the matrix $R$ using properties of orthogonal transformations?

We have $M = QR \Rightarrow Q^T M = Q^T QR = R$ since
$Q^TQ = I$. We are given the information to calculate $Q^TM$ and so we can find $R$.

**Question 3.** (Discussion)

Are the rows of an orthogonal matrix necessarily orthonormal? *Hint:* Consider the equation $QQ^T = I$ which holds for orthogonal transformations as $Q^T = Q^{-1}$.

1. Matrix multiplication is dot product of the rows of the first matrix with columns of the first. Hence $Q^T = I$ tells us exactly the rows are orthonormal.

2. Since $Q^T = Q^{-1}$ and we already know $Q^{-1}$ is orthogonal. Hence $Q^T$ orthogonal and it's columns are the rows of $Q$.

**Question 4.** Find an orthogonal transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} = T^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Hence this tells us the last column of $T^T$, or the last row of $T$.

$$T = \begin{pmatrix} a & b & c \\ d & e & f \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$$

From inspection, we can fill in the rest as:

$$T = \begin{pmatrix} -1/\sqrt{5} & 0 & 2/3 \\ 0 & -1/\sqrt{5} & 2/3 \\ 2/\sqrt{5} & 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$
\[ T = \begin{pmatrix} 0 & -\sqrt[3]{5} & \frac{2}{\sqrt[3]{5}} \\ \frac{2}{\sqrt[3]{5}} & 2 & \frac{1}{\sqrt[3]{5}} \end{pmatrix} \]

(You could also use Gram-Schmidt)

**Question 5.** (True/False, bonus questions) Determine which of the following are true or false.

(a) The equation \((AB)^T = A^T B^T\) holds for all \(n \times n\) matrices \(A\) and \(B\).

(b) The entries of an orthogonal matrix must be less than or equal to 1.

(c) If \(A\) and \(B\) are \(2 \times 2\) orthogonal matrices, then \(AB = BA\).

(d) If \(A\) is an invertible matrix then we have \((A^{-1})^T = (A^T)^{-1}\).

a) False. \((0 \ 1)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\)

and \((0 \ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\)

Hence \((\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\)

\((0 \ 1)^T (0 \ 0)^T = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\)

b) True.

c) False. Let \(A\) be a \(90^\circ\) anticlockwise rotation and \(B\) reflection in the \(X\)-axis.

Then \(A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) \(B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\)

\(AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\)
\[ BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

(A) True. If \( A \) is invertible, then \( AA^{-1} = I \)

\[ \Rightarrow (AA^{-1})^T = I \]

\[ \Rightarrow (A^{-1})^T A^T = I \]

so by definition, \((A^{-1})^T = (A^T)^{-1}\).