A set of vectors \( \vec{v}_1, \ldots, \vec{v}_n \) are linearly independent if \( \lambda_1 \vec{v}_1 + \cdots + \lambda_n \vec{v}_n = 0 \implies \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \).

In other words, \( \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \cdots & \frac{1}{n} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) has a unique solution. Otherwise, we say they are linearly dependent.

**Question 1.** Determine whether the vectors \( \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \\ 6 \end{pmatrix} \) are linearly independent or not.

Consider the coefficient matrix

\[
A = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix}
\]

we will row reduce,

\[
\begin{pmatrix} 1 & 1 & 4 \\ 2 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} - 2(\text{III})
\]

\[
\rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{RREF}(A).
\]

Since \( \text{RREF}(A) \) has a free column, \( Ax = b \) has no solution and the vectors are linearly dependent.
Question 2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the kernel of $A$, write $\begin{pmatrix} x \\ y \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$A \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} = 3.$$  Hence $\mathbf{v}_1 + 2\mathbf{v}_2 + 0\mathbf{v}_3 + 3\mathbf{v}_4 = 0$

$$\therefore \mathbf{v}_4 = \frac{1}{3} \mathbf{v}_1 + \frac{2}{3} \mathbf{v}_2$$

- From the previous question, we can see that elements of $\text{Ker}(A)$ give all the linear relations between the column vectors of $A$.

- A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis for a subspace $V$ of $\mathbb{R}^n$ if
  - They span $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
  - They are linearly independent.

- Given a collection $\mathbf{v}_1, \ldots, \mathbf{v}_n$, we can remove any vector that is a linear combination of the others without changing the span. If we remove enough this way, the collection eventually becomes linearly indep.

Question 3. From the following list of vectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ determine which vectors you can remove to make a basis of $\mathbb{R}^2$, and which ones you cannot.
We first want to find all linear combinations between these vectors. i.e. find the kernel of \( A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)

\[
\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

Therefore \( \ker(A) = \mathbb{R}^3 \).

These are all the possible linear combinations between the vectors. In particular, when \( t = 1 \), we get that

\[
\frac{1}{2} \overrightarrow{v}_1 + \frac{1}{2} \overrightarrow{v}_2 + \overrightarrow{v}_3 = 0.
\]

Hence \( \overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3 \) can be written as a linear combination of the others and so can be removed without changing the span. (and so form a basis)

\( \overrightarrow{v}_4 \) cannot be removed. The 0 in the kernel implies
\[ \vec{v}_4 \text{ cannot be removed. The } 0 \text{ in the kernel implies that } \vec{v}_4 \text{ cannot be written as a linear combination of the others.} \]

**Question 4.** (Discussion) Suppose that the column vectors \( \vec{v}_1, \ldots, \vec{v}_m \) are linearly related. Are the column vectors of \( \text{RREF}(\{\vec{v}_1, \ldots, \vec{v}_m\}) \) linearly related as well? If so, how are they related?

Let \( A = \left( \begin{array}{c} \vec{v}_1 \\ \vdots \\ \vec{v}_m \end{array} \right) \) then linear relations \( \lambda_1 \vec{v}_1 + \cdots + \lambda_m \vec{v}_m = 0 \) are given by the kernel \( \text{Ker}(A) = \text{Ker}(EA) \).

In particular, we see that if \( E \) is an elementary raw matrix, then \( \text{Ker} A = \text{Ker} EA \) since row operations don't change solution sets. That is, applying a row operation doesn't change how the columns are linearly related.

In particular, as \( \text{Ker} A = \text{Ker}(\text{RREF}(A)) \), the columns have the same linear relations.

**Example:**

Suppose we want to find a basis for the image of \( A = \left( \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array} \right) \). The image is the span of the columns. Hence we want to figure out which columns are redundant. We can find this out via the RREF:

\[
\left( \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array} \right) \rightarrow \left( \begin{array}{ccc} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{array} \right) \xrightarrow{x \cdot \frac{1}{2}} \left( \begin{array}{ccc} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{array} \right) \xrightarrow{x \cdot \frac{1}{3}}
\]


\[
\begin{pmatrix}
1 & 4 & 7 \\
0 & 1 & 2 \\
0 & 1 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{pmatrix} = \text{RREF}(A)
\]

Hence, if \( a_1, a_2, a_3 \) are the columns of \( \text{RREF}(A) \), then we can easily see that \( a_3 = 2a_2 - a_1 \). So if \( v_1, v_2, v_3 \) are the columns of \( A \), we have \( v_3 = 2v_2 - v_1 \) also.

So we can remove the last column to make a basis for the image. \( \text{Im}(A) = \text{span}\left(\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3/2 \\ 0 \\ 0 \end{pmatrix}\right) \). Note, just take the columns corresponding to those with leading 1's.

**Question 5.** Find a basis for the image and kernel of the matrix \( \begin{pmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{pmatrix} \)

\[
A = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{pmatrix}
\rightarrow
\begin{pmatrix} 1 & 2 & 9 \\ 0 & -7 & -28 \\ 0 & -15 & -60 \end{pmatrix}
\rightarrow
\begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}
\rightarrow
\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}
\]

Hence, the last column is linearly related to the first two. Hence \( \text{Im}(A) \) has basis given by \( \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3/2 \\ 0 \\ 0 \end{pmatrix} \).

(3) If \( a_1 \) that comprised in those will lead to 1's.)
\[
\begin{align*}
\text{Now, we find the kernel by } & \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \\
\Rightarrow \text{ solution of the form } & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3t \\ -4t \\ t \end{pmatrix} \in \mathbb{R} \\
\text{Hence } \ker A = \text{span}\left( \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \right) \text{ and this must be a basis for one vector.}
\end{align*}
\]

**Question 6.** Find a basis for the subspace of \( \mathbb{R}^3 \) defined by the equation

\[ x_1 + 2x_2 + 3x_3 = 0. \]

\[ x_1 + 2x_2 + 3x_3 = 0 \iff \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \\
\text{ie, we want } \ker A \text{ for } A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}. \text{ This is already in RREF, so solutions are } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2t - 3s \\ t \\ 5s \end{pmatrix} \in \mathbb{R} \\
\text{in particular, } \ker A = \left\{ t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}.
\]

\[ \text{and we see that the kernel is spanned by } \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ which are linearly independent and so form a basis.} \]